A & M #1.4: Helicon Waves

$$\vec{E}(t) = \vec{E}_0 e^{-i\omega t} \qquad \vec{E} \perp \vec{H} = H\hat{z}$$

a.) $E_y = \pm i E_x$.

E.O.M.: $m\vec{v}\,' = e\vec{E} + \frac{e}{c}\,\vec{v}\times\vec{H} - m\vec{v}/\tau \qquad (1)$

Let $v_\pm = v_x \pm i v_y$

Assume soln' of form $\vec{v}(t) = \vec{v}\, e^{-i\omega t} \longrightarrow (1)$ becomes

$$v_x{}' = \frac{e}{m}E_x - \frac{eH}{mc}v_y - v_x/\tau$$

$$v_y{}' = \frac{e}{m}E_y + \underbrace{\frac{eH}{mc}}_{\equiv \omega_c} v_x - v_y/\tau$$

$\longrightarrow$

$$v_\pm{}' = \frac{e}{m}E_\pm \mp i\omega_c v_\pm - v_\pm/\tau$$

$\longrightarrow$

$$v_\pm\left(-i\omega \pm i\omega_c + 1/\tau\right) = \frac{e}{m}E_\pm$$

So, $j_\pm = nev_\pm \longrightarrow$

$$j_\pm = \frac{\frac{ne^2\tau}{m}E_\pm}{1 - i(\omega \mp \omega_c)\tau} = \sigma_\pm E_\pm \qquad (2)$$

Obviously $j_z = 0$. Also (2) $+ E_y = \pm i E_x \rightarrow j_y = \pm i j_x$.

in circular polarization

σ is diagonal

16.) Maxwell's Eq's:

$$\begin{cases} \vec{\nabla} \times \vec{H} = 4\pi/c \, \vec{J} + 1/c \, \partial \vec{E}/\partial t & (0) \\ \vec{\nabla} \times \vec{E} = -1/c \, \partial \vec{H}/\partial t & (1) \end{cases}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla} \cdot \vec{E}}) - \nabla^2 E \quad (2)$$

Combining (1) & (2) gives

$$-\nabla^2 \vec{E} = \vec{\nabla} \times (-1/c \, \partial \vec{H}/\partial t)$$

$$= -1/c \, \vec{\nabla} \times -i\omega \vec{H}$$

$$(0) \longrightarrow \quad = i\omega/c \left[4\pi/c \, \vec{J} + 1/c \, \partial \vec{E}/\partial t \right] \quad (3)$$

Assume a soln' of the form $E_x = E_0 e^{i(kz - \omega t)}$

$E_y = \pm i E_x$; $E_z = 0$.

$(3) \longrightarrow$

$$+k^2 E_x = \frac{i\omega}{c} \left[4\pi/c \, J_x - i\omega/c \, E_x \right]$$

$$k^2 c^2 E_x = \omega^2 \Big[\underbrace{\frac{4\pi}{i\omega} \frac{J_x}{E_x} + 1}_{\equiv \epsilon(\omega)} \Big] E_x \quad (4)$$

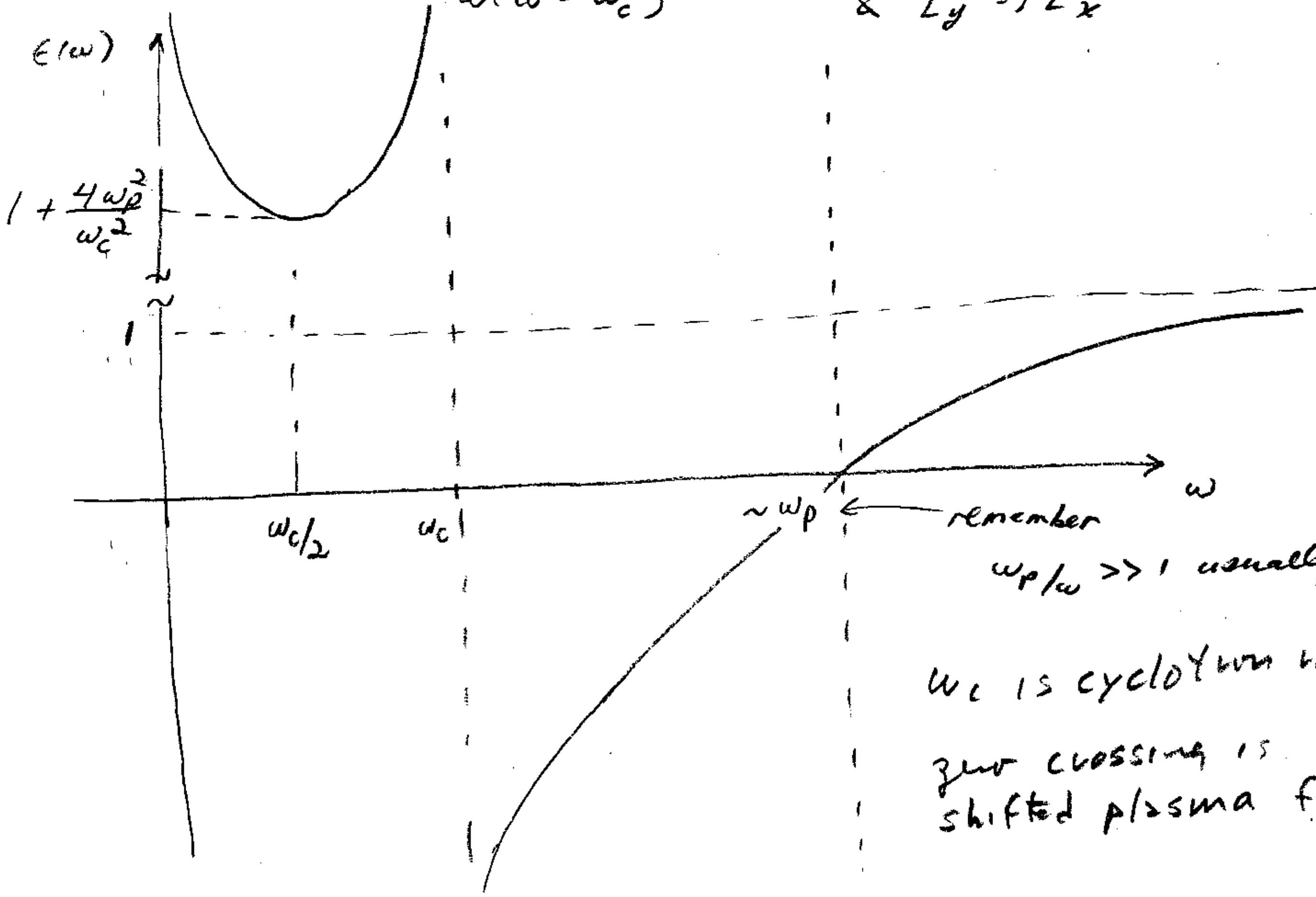
Notice from (a) that $\epsilon(\omega)$ can be written:

$$\epsilon(\omega) = \frac{4\pi}{i\omega} \frac{\sigma_0}{1 - i(\omega \mp \omega_c)\tau} + 1$$

$$= 1 - \frac{\omega_p^2}{\omega} \left(\frac{1}{i/\tau + \omega \mp \omega_c} \right) \quad \text{where } \omega_p^2 = 4\pi \sigma_0/\tau = \frac{4\pi n e^2}{m}$$

Thus, (4) has a soln' only when $k^2 c^2 = \omega^2 \epsilon(\omega)$.

1c.) $\epsilon(\omega) = 1 - \dfrac{\omega_p^2}{\omega(\omega - \omega_c)}$ || from (4) $-\omega-\omega_c\tau >> 1$ & $E_y = iE_x$



remember $\omega_p/\omega >> 1$ usually.

ω_c is cyclotron resonance

zero crossing is shifted plasma freq.

when $\omega > \omega_p$ or $\omega < \omega_c$, $\epsilon(\omega)$ is (+) and can therefore satisfy (4).

$\omega_p >> \omega << \omega_c$

1d.) assume $\omega << \omega_c \rightarrow \epsilon(\omega) \sim 1 + \dfrac{\omega_p^2}{\omega\omega_c} \simeq \dfrac{\omega_p^2}{\omega\omega_c}$

Thus (4) is

$$k^2c^2E_x = \omega\,\omega_p^2/\omega_c \rightarrow \boxed{\omega = \frac{k^2c^2\,\omega_c}{\omega_p^2}}$$ Helicon freq.

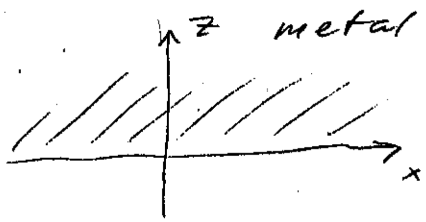
Since, $\omega_p \sim 10^{16}\,s^{-1}$
$\omega_c = eH/mc \sim 10^{+11}\,s^{-1}$
$ck \sim 10^{11}$

$\Big\}\ \omega \sim 10\,s^{-1}$

Problem #1.5 | Surface Plasmons

$$\vec{E} = (A\vec{e}_x + B\vec{e}_z)\, e^{iqx - kz}, \quad z>0$$

$$\vec{E} = (C\vec{e}_x + D\vec{e}_z)\, e^{iqx + k'z}, \quad z<0$$



(a) $\varepsilon(\omega) = 1 + \frac{4\pi i \sigma}{\omega}$

$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$; $\sigma_0 = \frac{ne^2\tau}{m}$

$\vec{E}$ satisfies to the Maxwell equation:

$$-\nabla^2 \vec{E} = \frac{\omega^2}{c^2}\varepsilon(\omega)\vec{E} \qquad (1.34)$$

① $z>0$ $\quad -(-q^2 + k^2) = \frac{\omega^2}{c^2}\varepsilon(\omega)$

$$q^2 - k^2 = \frac{\omega^2}{c^2}\left(1 + \frac{4\pi i}{\omega}\,\frac{\frac{e^2 n\tau}{m}}{1 - i\omega\tau}\right)$$

$$\boxed{(q^2 - k^2)c^2 = \omega^2 + \frac{i\,\omega_{pl}^2\,\omega\tau}{1 - i\omega\tau}} \quad - (1)$$

$\nabla\vec{E} \sim \operatorname{div} \mathcal{g} = 0$ for $z>0$; $z<0$

$$\boxed{(Aiq - Bk) = 0} \quad - (2)$$

(2) $z<0$

$$-\nabla^2\vec{E} = \frac{\omega^2}{c^2}\vec{E}\,(\varepsilon(\omega) - 1))$$

$$(q^2 - k'^2) = \frac{\omega^2}{c^2}$$

$$\boxed{c^2(q^2 - (k')^2) = \omega^2} \quad - (3)$$

$$\boxed{C\,iq + D\cdot k' = 0} \quad - (4)$$

E_x – continuous $\Rightarrow$ $\boxed{A = C}$

εE_z – continuous $\boxed{\varepsilon B = D}$

Then: $Bk = -Dk' \Rightarrow Bk = -\varepsilon B k'$

$\boxed{k = -\varepsilon k'}$ (*)

Three equations:

$$\begin{cases} (q^2 - k^2)c^2 = \omega^2 + \frac{i\omega_{pl}^2\omega\tau}{1 - i\omega\tau} & (1) \\ k = -\left(1 + \frac{4\pi i}{\omega}\,\frac{\sigma_0}{1 - i\omega\tau}\right)k' & (2) \\ q^2 - (k')^2 = \frac{\omega^2}{c^2} & (3) \end{cases}$$

(b) $\omega\tau \gg 1$ $\quad \varepsilon(\omega) = 1 + \frac{i\omega_{pl}^2\tau}{\omega(1 - i\omega\tau)} \approx 1 - \frac{\omega_{pl}^2}{\omega^2}$

$$\begin{cases} (q^2 - k^2)c^2 = \omega^2 - \omega_{pl}^2 \\ k = -\left(1 - \frac{\omega_{pl}^2}{\omega^2}\right)k' \\ c(q^2 - (k')^2) = \omega^2 \end{cases} \qquad k' = -\frac{\omega^2}{\omega^2 - \omega_{pl}^2}\,k$$

$$q^2 - \frac{\omega^4}{(\omega^2 - \omega_{pl}^2)^2}\,k^2 = \frac{\omega^2}{c^2}$$

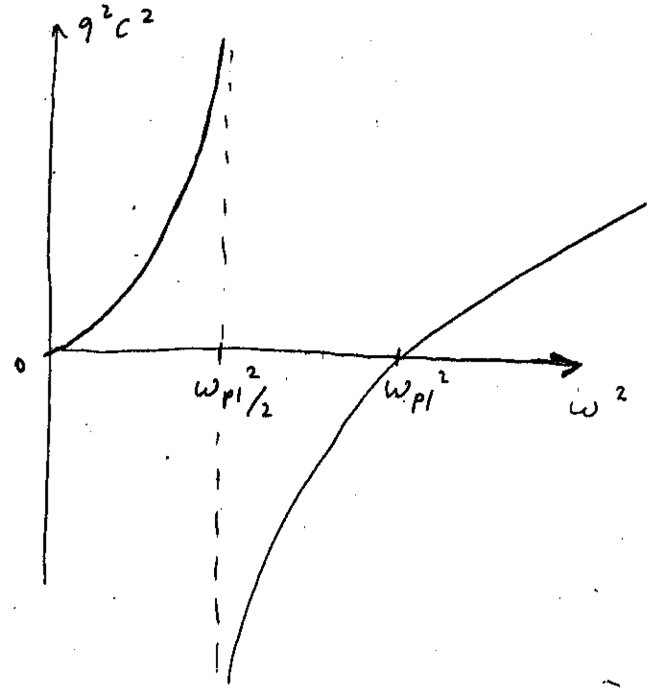
$$k^2 = \frac{(\omega^2 - \omega_{pl}^2)^2}{\omega^4}\left(q^2 - \frac{\omega^2}{c^2}\right)$$

$$q^2c^2 - (q^2c^2 - \omega^2)\frac{(\omega^2 - \omega_{pl}^2)^2}{\omega^4} = \omega^2 - \omega_{pl}^2$$

$$q^2c^2\left(1 - \frac{(\omega^2 - \omega_{pl}^2)^2}{\omega^4}\right) = \omega^2 - \omega_{pl}^2 - \frac{(\omega^2 - \omega_{pl}^2)^2}{\omega^2}$$

$$q^2c^2 = \omega^2\,\frac{\omega^4 - \omega^2\omega_{pl}^2 - (\omega^2 - \omega_{pl}^2)^2}{\cancel{\omega^4} - \cancel{\omega^4} + 2\omega^2\omega_{pl}^2 - \omega_{pl}^4} = \omega^2\,\frac{(\omega^2 - \omega_{pl}^2)(\omega^2 - \omega^2 + \omega_{pl}^2)}{(2\omega^2 - \omega_{pl}^2)\,\omega_{pl}^2}$$

$q^2c^2 = \frac{\ldots}{(2\omega^2-\omega_{pl}^2)}$



(c) $qc \gg \omega$ from the graph: $\omega \approx \frac{\omega_{pl}}{\sqrt{2}}$

$$K = -\left(1-\frac{2\omega_{pl}^2}{\omega_{pl}^2}\right)K' = K'$$

$$K^2c^2 = q^2c^2-\omega^2 \simeq q^2c^2 \Rightarrow$$

$$\Rightarrow \boxed{K = K' \simeq q}$$

q is a large number $\Rightarrow$ the wave is localized near the surface.

Polarization:

$\frac{C}{D} = \frac{E_x(z<0)}{E_z(z<0)}$; $\frac{A}{B} = \frac{E_x(z>0)}{E_z(z>0)}$

$\frac{A}{B} = \frac{\varepsilon C}{D}$; $\varepsilon = 1-\frac{\omega_{pl}^2}{\omega^2} = -1$

$\frac{A}{B} = -\frac{C}{D}$; $\frac{A}{B} = \frac{K}{iq} \simeq \frac{1}{i}$; $\frac{C}{D} = -\frac{K'}{iq} = -\frac{1}{i}$ $\boxed{2.2}$

$\boxed{\frac{A}{B} = i\ ;\ \frac{C}{D} = +i}$ For $z>0$ or $z<0$ there is a circular polarization (right or left)

Problem 2.1

(a) $n = \frac{2}{(2\pi)^2}\int dk_x\, dk_y = \frac{2\cdot(2\pi)}{(2\pi)^2}\int_0^{k_F} k\cdot dk = \frac{k_F^2}{2\pi}$

(b) $\pi r_s^2 = \frac{1}{n}$; $\pi r_s^2 = \frac{2\pi}{k_F^2}$; $\boxed{r_s = \frac{\sqrt{2}}{k_F}}$

(c) $g(\varepsilon)d\varepsilon = \frac{2\cdot 2\pi}{(2\pi)^2}k\,dk$

$k\,dk = m\,d\varepsilon$ $(\varepsilon>0)$

$k\,dk = 0$ $(\varepsilon<0)$

$$g(\varepsilon) = \begin{cases} \frac{m}{\pi} & \varepsilon>0 \\ 0 & \varepsilon<0 \end{cases}$$

In units: $\boxed{\hbar = 1}$

(d) $n = \int_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon)\, f(\varepsilon)$

Let's introduce function $K(\varepsilon) = \int_{-\infty}^{\varepsilon} d\varepsilon'\, g(\varepsilon)$

Then $n = \int_{-\infty}^{\infty} dK(\varepsilon)\cdot f(\varepsilon) = \underbrace{K(\varepsilon)f(\varepsilon)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} d\varepsilon\, f'(\varepsilon)\, K(\varepsilon)$

$K(-\infty) = 0$; $f(\varepsilon=\infty) = 0$

$$n = -\int_{-\infty}^{\infty} d\varepsilon\, f'(\varepsilon)K(\varepsilon) = -\int_{-\infty}^{\infty} d\varepsilon\, f'(\varepsilon)\left(K(\mu)+\sum_{n=1}^{\infty}\frac{(\varepsilon-\mu)^n}{n!}\frac{d^nK}{d\varepsilon^n}\right.$$

$$K(\mu) = \int_{-\infty}^{\mu} d\varepsilon\, g(\varepsilon) = \frac{m}{\pi}\mu = n$$

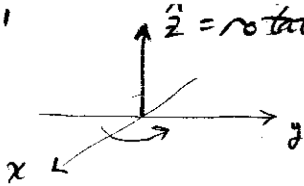
$$K'(\varepsilon) = g(\varepsilon) = \frac{m}{\pi}\theta(\varepsilon)$$

$K''(\varepsilon) \ldots \quad K^{(n)}(\varepsilon) = \ldots$

3) In general,

$$\vec{J} = \hat{\sigma}\,\vec{E}$$

FIG. 1

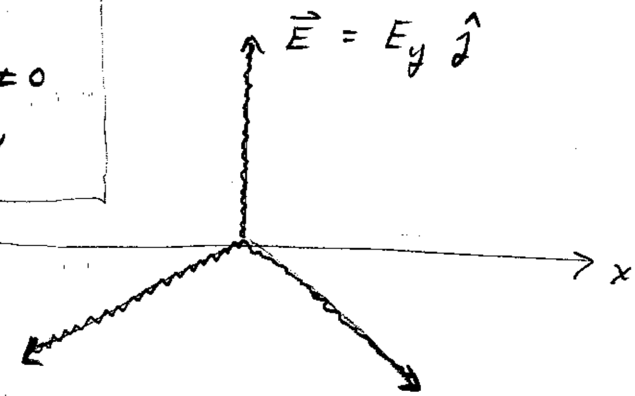


Let $\vec{z} = \hat{k}$ be the rotation axis so we will only be concerned with σ in the xy-plane. Then

$$\vec{J} = (\sigma_{xx} E_x + \sigma_{xy} E_y)\,\hat{\imath} + (\sigma_{yx} E_x + \sigma_{yy} E_y)\,\hat{\jmath} \quad (1)$$

Consider the 3-fold symmetric case and assume $\vec{E} = E_y\,\hat{\jmath}$ is in the y-direction. In this case, if $\sigma_{xy} \neq 0$ then, according to (1), the current $\vec{J}$ will have a component in the x-direction. However, inversion symmetry $x \to -x$ of the problem (see Fig. 2) demands $\vec{J}$ has the same component in the $-x$ direction, which requires $\sigma_{xy} = -\sigma_{xy} \Rightarrow \sigma_{xy} = 0$. Likewise for $\sigma_{yx} = 0$.

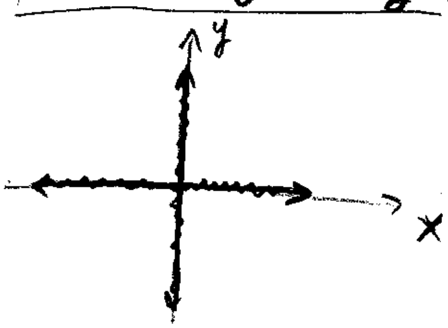
FIG. 2



NOTICE $x \to -x$ SYMMETRY.

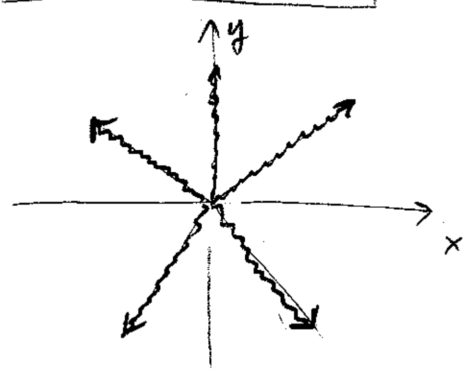
The argument is the same for all higher order symmetries. Just change FIG. 2 as follows:

4-fold symmetry



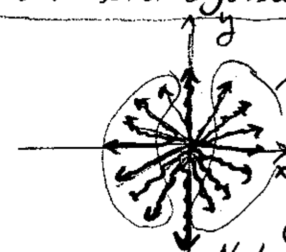
<Note: in this case $x \to y$ symmetry demands $\sigma \propto \mathbb{1}$>

5-fold symmetry



<as before, notice $x \to -x$ symmetry>

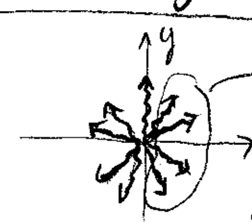
EVEN-fold symmetry



even # of symmetry axis on each side plus 2 more symmetry axis @ $\pm y$.

Note: $x \to -x$ symmetry

ODD-fold symmetry



even # of symmetry ax on both sides plus on symmetry axis @ $+y$.

Notice $x \to -x$ symmetry ✓

8 cont.) So we have convinced ourselves that $\hat{\sigma}$ should be diagonal. It turns out we can show that $\hat{\sigma} \propto \mathbb{1}$!
Let $\theta_n = 360/n \ni n = 1, 2, 3, \ldots$ Then if our system is n-fold degenerate we must have

$$\hat{\sigma} = \hat{R}_{-\theta_n} \hat{\sigma} \hat{R}_{\theta_n} \qquad \Big\| \quad \hat{R}_{\theta_n} \equiv \text{rotation by } \theta_n$$

$$\begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{pmatrix} = \begin{pmatrix} \cos(-\theta_n) & \sin(-\theta_n) \\ -\sin(-\theta_n) & \cos(-\theta_n) \end{pmatrix} \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{pmatrix} \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & \cos\theta_n \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix} \begin{pmatrix} \sigma_{xx}\cos\theta_n & \sigma_{xx}\sin\theta_n \\ -\sigma_{yy}\sin\theta_n & \sigma_{yy}\cos\theta_n \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \sigma_{xx}\cos^2\theta_n + \sigma_{yy}\sin^2\theta_n & \sigma_{xx}\cos\theta_n\sin\theta_n - \sigma_{yy}\cos\theta_n\sin\theta_n \\ \hline \sigma_{xx}\sin\theta_n\cos\theta_n - \sigma_{yy}\sin\theta_n\cos\theta_n & \sigma_{xx}\sin^2\theta_n + \sigma_{yy}\cos^2\theta_n \end{array}\right)$$

Obviously, this can only be satisfied when $\sigma_{xx} = \sigma_{yy}$.

Thus, $\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{xx} \end{pmatrix} \propto \mathbb{1}$. $\blacksquare$

$$\vec{J} = \hat{\sigma}\vec{E}$$

$$\underbrace{\hat{\sigma}^{-1}}_{\hat{\rho}}\vec{J} = \vec{E}$$

$$\longrightarrow \hat{\rho} \equiv \hat{\sigma}^{-1}$$

$\rightarrow$ This implies that $\rho_{xx} \neq 1/\sigma_{xx}$! Afterall, consider the 2D-case:

$$\hat{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$$

$$\hat{\rho} \equiv \hat{\sigma}^{-1} = \frac{1}{|\hat{\sigma}|}\begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix}$$

$$\longrightarrow \rho_{xx}^{-1} = \sigma_{xx}/|\hat{\sigma}|$$

(4) Let's consider equation of motion for electron under "force" $\vec{\mathcal{E}} = \vec{\mathcal{E}}_0 e^{-i\omega t}$; linear restoring force

$$d\vec{p}/dt = -\vec{p}/\tau - e\vec{\mathcal{E}} - \kappa\vec{x}, \quad \text{where } \vec{p} = m\dot{\vec{x}}$$

Looking for a solution in form $\vec{x} = \vec{x}_0 e^{-i\omega t}$, we have:

$$-\vec{x}_0\omega^2 m = i\omega/\tau\, m\vec{x}_0 - e\vec{\mathcal{E}}_0 - \kappa\vec{x}_0 \Rightarrow$$

$$\Rightarrow \quad \vec{x}_0(\kappa - m\omega^2 - i\omega/\tau\, m) = -e\vec{\mathcal{E}}_0$$

$$\vec{x}_0 = \frac{-e\vec{\mathcal{E}}_0/m}{\omega_0^2 - \omega^2 - i\omega/\tau}, \text{ where we introduced } \omega_0^2 = \frac{\kappa}{m}$$

For a current density we get:

$$\vec{j} = -ne\vec{v} = \frac{ne^2}{m}\cdot\frac{-i\omega}{\omega_0^2 - \omega^2 - i\omega/\tau}\vec{\mathcal{E}}_0$$

For σ we have: $\sigma(\omega) = \frac{ne^2}{m}\frac{-i\omega}{\omega_0^2 - \omega^2 - i\omega/\tau}$

At large frequencies' limit ($\omega \gg \omega_0, 1/\tau$),

we get $\sigma = \frac{ne^2 i}{m\omega} \Rightarrow \sigma_2(\omega) = ne^2/m\omega$

$$\sigma(\omega) = \frac{ne^2 i}{m\omega}\cdot\frac{1}{1 + i/\omega\tau} = \frac{ne^2 i}{m\omega}\cdot\frac{\omega\tau(\omega\tau - i)}{(\omega\tau)^2 + 1} \Rightarrow$$

$$\Rightarrow \sigma_1(\omega) = \frac{ne^2}{m\omega}\frac{\omega\tau}{(\omega\tau)^2 + 1} \approx \frac{ne^2}{m\omega^2\tau}$$

Now we can check sum rule for $\sigma_2(\omega)$:

$$\sigma_2(\omega) = -\frac{2\omega}{\pi}\cdot\frac{ne^2\tau}{m}\int_0^{+\infty}\frac{d\omega'\,\sigma_1(\omega')}{\omega'^2 - \omega^2} \approx$$

$$\approx +\frac{2\omega ne^2\tau}{\pi m}\cdot\frac{1}{\omega^2}\int_0^{+\infty}\frac{d\omega'}{(\omega'\tau)^2 + 1} = \frac{2\omega ne^2\tau}{\pi m\omega^2}\cdot\frac{1}{\tau}\cdot\frac{\pi}{2} = \frac{ne^2}{m\omega}$$

5) Kramer's Kronig Relations.

$$\begin{cases} \sigma_1(\omega) = {}^2/_\pi \, ⨍_0^\infty \dfrac{\omega' \sigma_2(\omega')\,d\omega'}{\omega'^2 - \omega^2} & \text{(1a} \\ \sigma_2(\omega) = -{}^{2\omega}/_\pi \, ⨍_0^\infty \dfrac{\sigma_1(\omega')\,d\omega'}{\omega'^2 - \omega^2} & \text{(1b} \end{cases} \quad \ni \; ⨍ \equiv \text{principle value}$$

The Drude conductivity is: $\sigma = \dfrac{ne^2\tau/m}{1 - i\omega\tau} \equiv \sigma_0/(\gamma - i\omega\tau)$

$\longrightarrow$

$$\sigma = \sigma \frac{1 + i\omega\tau}{1 + i\omega\tau} = \frac{\sigma_0(1 + i\omega\tau)}{1 + (\omega\tau)^2} \Longrightarrow \begin{cases} \sigma_1 = \sigma_0/(1 + \omega^2\tau^2) \\ \sigma_2 = \sigma_0\omega\tau/(1 + \omega^2\tau^2) \end{cases}$$

Let's show these satisfy (1). Begin with (1a):

$$\sigma_1 = {}^2/_\pi ⨍_0^\infty \frac{\omega'\sigma_2(\omega')d\omega'}{\omega'^2 - \omega^2} = {}^2/_\pi ⨍_0^\infty \frac{\omega'}{\omega'^2 - \omega^2} \frac{\sigma_0\omega'\tau}{1 + \omega'^2\tau^2}\,d\omega'$$

$$= {}^2/_\pi \, {}^{\sigma_0}/_\tau ⨍_0^\infty \frac{1}{\omega'^2 - \omega^2} \frac{\omega'^2}{\omega'^2 + (1/\tau)^2}\,d\omega'$$

$$= \frac{2}{\tau\pi} \frac{\sigma_0}{\omega^2 + (1/\tau)^2} \int_0^\infty \underbrace{\frac{\omega^2}{\omega'^2 - \omega^2}}_{0} + \underbrace{\frac{(1/\tau)^2}{\omega'^2 + (1/\tau)^2}}_{\pi/2\,(1/\tau)}\,d\omega$$

$$= \frac{\sigma_0}{1 + (\omega\tau)^2}$$

$$= \sigma_1 \; \checkmark$$

Likewise for (1b):

$$\sigma_2 = -{}^{2\omega}/_\pi ⨍_0^\infty \frac{\sigma_1(\omega')d\omega'}{\omega'^2 - \omega^2} = -\frac{2\omega}{\pi\tau^2}\sigma_0 \int_0^\infty \frac{1}{\omega'^2 + (1/\tau)^2} \frac{1}{\omega'^2 - \omega^2}\,d\omega'$$

$$= \frac{-2}{\pi\tau^2}\,\omega\sigma_0 \frac{-1}{\omega^2 + (1/\tau)^2} \underbrace{\int_0^\infty \frac{1}{\omega'^2 + (1/\tau)^2}}_{= \pi/2\,\tau} - \underbrace{\frac{1}{\omega'^2 - \omega^2}}_{= 0}\,d$$

$$= \frac{\sigma_0\omega\tau}{(\omega\tau)^2 + 1}$$

$$= \sigma_2 \; \checkmark$$

6 cont.) The sum rule says $\int_0^\infty \sigma \, d\omega = $ constant.

(b.) Let's see if this is true in the high frequency limit

Here $\sigma_2 = \dfrac{\sigma_0 \omega \tau}{(1+\omega^2\tau^2)} \longrightarrow \dfrac{\sigma_0}{\omega\tau} = \dfrac{ne^2}{m\omega}$

Thus, according to (16),

$$\sigma_2 = ne^2/m\omega = -\frac{2\omega}{\pi}\int_0^\infty \frac{\sigma_1(\omega')\,d\omega'}{\omega'^2-\omega^2}$$

$$\approx +2/\omega\pi \int_0^\infty \sigma_1(\omega')\,d\omega' \quad \text{// high freq. limit}$$

$\longrightarrow$

$$\int_0^\infty \sigma_1(\omega')\,d\omega' = \frac{\pi n e^2}{2m} = \text{const.} \ \checkmark$$

(c.) For Cu @ T_R, $\omega_p \sim 1.6 \times 10^{16}/s$ TABLES A & M 1.1

$\tau \sim 2.7 \times 10^{-14}\,s$ A & M 1.3

$\omega \sim 10^{14}/s$

In this case $\omega_p\tau >> 1$ & $\omega_p >> \omega$. In these cases, it can be shown (not here) that $A(\omega)$ satisfies

$$A(\omega) = \frac{2\sqrt{2}}{\omega_p\tau}\,\omega\tau\left\{\sqrt{1+(1/\omega\tau)^2}-1\right\}^{1/2}$$

$\longrightarrow$

MOTT-ZENER: $\omega\tau >> 1 \longrightarrow A(\omega) \cong \dfrac{2\sqrt{2}}{\omega_p\tau}\,\omega\tau\left\{1+\tfrac{1}{2}(1/\omega\tau)^2-1\right\}^{1/2} = \dfrac{2}{\omega_p\tau} \sim 10^{-2}$

HAGEN-RUBENS: $\omega\tau << 1 \longrightarrow A(\omega) \approx \dfrac{2\sqrt{2}}{\omega_p\tau}\,\omega\tau\left\{\dfrac{1}{\omega\tau}\right\}^{1/2} \sim \dfrac{2\sqrt{2}}{\omega_p\tau}(\omega\tau)^{1/2} \sim \dfrac{2}{\omega_p}\sqrt{\dfrac{2\omega}{\tau}} \sim 10^{-1}$

Let's see if the Drude result agrees. We know

$$R = \left|\frac{\sqrt{\epsilon}-1}{\sqrt{\epsilon}+1}\right|^2 = \left|\frac{1-1/\sqrt{\epsilon}}{1+1/\sqrt{\epsilon}}\right|^2$$

- If $\omega\tau >> 1 \longrightarrow \epsilon \cong 1 - (\omega_p/\omega)^2\left[\dfrac{1}{i/\omega\tau + 1}\right] \approx -\left(\dfrac{\omega_p}{\omega}\right)^2\left\{\dfrac{1}{1+i/\omega\tau}\right\} >> 1$

$\longrightarrow 1/\sqrt{\epsilon} \sim -i\left(\dfrac{\omega}{\omega_p}\right)\left[1+\dfrac{i}{2\omega\tau}\right]$

$\longrightarrow R \cong 1 - 4\,\mathrm{Re}(1/\sqrt{\epsilon})$

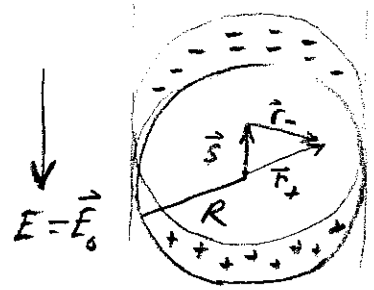
$\sim 1 - 2/\omega_p\tau \Rightarrow A = 2/\omega_p\tau \sim 10^{-2}$

- If $\omega\tau << 1 \longrightarrow \epsilon \sim 1 + i\,\omega_p^2\tau/\omega \overset{\omega_p>>\omega}{\sim} i\,\omega_p^2\tau/\omega \rightarrow \sqrt{\epsilon} = \underbrace{\omega_p\sqrt{\tau/2\omega}}_{\equiv B}(1+i)$

$\longrightarrow R \sim \left|\dfrac{B(1+i)-1}{B(1+i)+1}\right|^2 \cong 1 - \dfrac{2}{B} \sim 1 - \dfrac{2\sqrt{2}}{\omega_p}\sqrt{\omega/\tau} \Rightarrow A = \dfrac{2}{\omega_p}\sqrt{\dfrac{2\omega}{\tau}}$

$\longrightarrow A \sim 10^{-2}$

6a) We will solve this iteratively. First we find the polarization from $\vec{E} = \vec{E}_0$



The negative charge is shifted up a little (in this picture), while the positive charge is shifted down a little. The positive sphere has a charge $+Q$ while the negative sphere has a charge $-Q$ (net charge $= 0$). The $\vec{E}$-field @ any point inside the sphere is a sum of the $\vec{E}$-field from the (+) sphere and the (−) sphere, $\vec{E}_+$ & $\vec{E}_-$, respectively

$$\vec{E}_1 = \vec{E}_+ + \vec{E}_- \qquad (1)$$

Using Gauss' law we can find $\vec{E}_\pm$ pretty easily:

$$\int \vec{E}_\pm \cdot \vec{da} = 4\pi q_{enc}$$

$$\vec{E}_\pm \, 4\pi \vec{r}^2 = 4\pi\left(\pm \frac{\vec{r}^3}{R^3} Q\right) \quad \Big\| \; q_{enc} = \frac{4/3\, \pi r^3}{4/3\, \pi R^3} Q$$

$$\vec{E}_\pm = \pm \frac{\vec{r}_\pm}{R^3} Q$$

(1) $\longrightarrow$

$$\vec{E}_1 = \frac{Q}{R^3}(\vec{r}_+ - \vec{r}_-)$$

$$= \frac{Q}{R^3}\vec{s}$$

$$= \frac{\vec{p}}{R^3} \quad \Big\| \; \vec{p} = Q\vec{s} = \text{polarization}$$

$\therefore \; \vec{P}_1 = \vec{P}_1/V = \vec{P}_1/(4/3\, \pi R^3)$, so $\vec{E}_1 \equiv \frac{4\pi}{3}\vec{P}_1$.

$$\equiv -\frac{4\pi}{3}\chi \vec{E}_0 \qquad (2)$$

Thus far we have $\vec{F} = \vec{E}_0 + \vec{E}_1 = E_0(1 - 4\pi\chi/3)$

However, the polarized field $\frac{4\pi}{3}\chi\vec{E}_0$ will further polarize the sphere, producing $\vec{E}_2$:

$$\vec{E}_2 = -4\pi/3\ \chi\ \vec{E}_1 \qquad \| \text{ from (2)}$$

and so on ...

$$\vec{E}_n = 4\pi/3\ \chi\ \vec{E}_{n-1} = (-4\pi/3\ \chi)^n\ \vec{E}_0$$

Thus, the final $\vec{E}$-field inside the sphere is

$$\vec{E}_{TOT} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \dots$$

$$= \sum_n (-4\pi/3\ \chi)^n\ \vec{E}_0$$

$$= \left(\frac{1}{1 + 4\pi/3\ \chi}\right)\vec{E}_0 \qquad \| \text{ Geometric Series}$$

$$\boxed{\vec{P}_{TOT} = \chi\ \vec{E}_{TOT} = \frac{\chi}{1 + 4\pi/3\ \chi}\ \vec{E}_0} \qquad (3)$$

b.) When $\omega >> 1/\tau$, $\epsilon(\omega) \to 1 - \omega_p^2/\omega^2$. Since $\epsilon(\omega) = 1 + 4\pi\chi(\omega)$ we have

$$\chi(\omega) = (\epsilon(\omega) - 1)/4\pi = -\omega_p^2/4\pi\omega^2$$

c) Resonance occurs when (3) blows up, or when

$$1 + 4\pi/3\ \chi = 0$$

$$1 + \cancel{4\pi}/3\left(-\omega_p^2/\cancel{4\pi}\omega_r^2\right) = 0$$

$$\boxed{\omega_r^2 = \omega_p^2/3}$$