4. $\boxed{1}$ (a) (2-D electron gas)

$\to n = \frac{2}{(2\pi)^2} \frac{k_F^2 2\pi}{2} \to k_F = \sqrt{2\pi n} = 1.37\ 10^6\ cm^{-1}$

$\to \epsilon_F = \frac{\hbar^2 k_F^2}{2m^*} = \frac{10^{-68} \cdot 1.9\ 10^{16}}{2(0.07)\ 9.11\ 10^{-31}} \simeq 1.5\ 10^{-21} J = 10^{-2} eV$

$\to v_F = \frac{\hbar k_F}{m^*} = 2.1\ 10^5\ m/s$

$\to \mu = \frac{e\tau}{m^*} \to \tau = \frac{m^* \mu}{e} = \frac{0.07 \cdot 9.11 \cdot 10^{-31} \cdot 2\ 10^2}{1.6\ 10^{-19}} = 7.9\ 10^{-11}\ s$

$l = v_F \tau = 1.7\ 10^{-5}\ m$

$\to \Delta E = \frac{\hbar}{\tau} \frac{1}{k_B} = 9\ 10^{-2}\ K$. (using $\Delta E \tau \sim \hbar$ and $k_B T \sim \Delta E$)

(b)

$$n = \int_0^\infty d\epsilon\ g(\epsilon) \frac{1}{\exp[-\beta(\mu - \epsilon)] + 1}$$

$g(\epsilon)$ is a const. in 2D and $\epsilon_l = \hbar\omega_c (l + 1/2)$ Landau level energies. Suppose $\beta\hbar\omega_c \gg 1$ with every Landau level with g_0 electrons. $(l-1)$ levels occupied completly

$\therefore\ n = g_0(l-1) + \frac{g_0}{\exp\left(-\beta(\mu - \epsilon_l) + 1\right)}$ where $g_0 = \frac{eS}{hc} H \propto H$

$$\frac{g_0 l - n}{n - g_0(l-1)} = \exp\left[-\beta(\mu - \epsilon_l) + 1\right]$$

$$\boxed{\mu = \epsilon_l + \frac{1}{\beta} \ln\left(\frac{n - g_0(l-1)}{g_0 l - n}\right)}$$

$\boxed{\text{if } T = 0 \to \epsilon_F = \epsilon_l}$

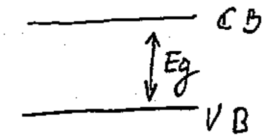
4. $\boxed{1-C}$ $B = 10\,T$

$$l_0 = \left(\frac{\hbar}{eB}\right)^{1/2} = \left(\frac{10^{-34}}{1.6\,10^{-9}\,10}\right)^{1/2} = 8\,10^{-9}\,m$$

$$\tan\theta = \omega_c \tau = \frac{eB\tau}{m} = 2\,10^{3}.$$

i.e. $\theta \approx 90^\circ \rightarrow$ Hall current is much stronger than the current along the electric field: that happens in very strong magnetic fields.

5 (3)



(a) $(\vec{B}=0) \to \vec{H}=0$ (no magnetic field)

$$H = \frac{p^2}{2m} + V(r) \quad , \ m = \text{free electron mass.}$$

$|\psi\rangle \to$ Bloch $\to$ $|\psi\rangle \propto e^{i\vec{k}\cdot\vec{r}}\, u_{n,\vec{k}}$

$\to$ use $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$ with $H\,|\psi_{\vec{k},n}\rangle = E_n(\vec{k})\,|\psi_{\vec{k},n}\rangle \to \Big(\underbrace{\frac{p^2}{2m}+V(r)}_{H_0} + \underbrace{\frac{\hbar}{m}\vec{k}\cdot\vec{p} + \frac{\hbar^2 k^2}{2m}}_{H'}\Big) u_{n,\vec{k}} = E\,u_{n,\vec{k}}$

About $\vec{k}=0$ H' is a perturbation to H_0

$H_0 u_{n,0} = E_n u_{n,0}$ (VB $\to n=1$; CB $\to n=2$). Two ways of solving for eigenvalues:

(1) Use Perturbation theory formula, (2) This is just a 2 level system, write matrix and diagonalize this matrix (see C. COHEN-TANNOUDJI p. 405). Using the 2^{nd}:

$\to u_{n,\vec{k}} = \sum_m a_m(\vec{k})\, u_{m,0}$, $\langle u_n|\vec{p}|u_m\rangle = \vec{q}_{n,m}\,\delta_{n,m}$, $(H_0+H')\,u_{n,\vec{k}} = E_n(\vec{k})\,u_{n,\vec{k}}$, $E_2(\vec{k}) = E_1(\vec{k}) = E(\vec{k})$

$$\begin{bmatrix} E_2 + \frac{\hbar^2 k^2}{2m} - E(\vec{k}) & \frac{\hbar}{m}\vec{k}\cdot\vec{q}_{2,1} \\ \frac{\hbar}{m}\vec{k}\cdot\vec{q}_{12} & E_1 + \frac{\hbar^2 k^2}{2m} - E(\vec{k}) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \ , \quad \vec{q}_{2,1} = \vec{q}_{1,2} = \vec{q} \ : \ \text{Symmetric here (no } \vec{B})$$

$E_g \equiv E_2 - E_1$

$E(\vec{k}) = \frac{1}{2}(E_2+E_1) + \frac{\hbar^2 k^2}{2m} \pm \frac{E_g}{2}\sqrt{1+\frac{4\hbar^2}{m^2 E_g^2}(\vec{k}\cdot\vec{q})^2}$; Effective mass is such $\to E_n(\vec{k}) - E_n(0) \equiv \frac{\hbar^2}{2}\sum_{\alpha,\beta}\left(\frac{1}{m^*}\right)_{\alpha\beta} k_\alpha k_\beta$

$\downarrow$

$E_2(\vec{k}) - E_2 = -\frac{1}{2}(E_2+E_1+E_2-E_1) + \frac{1}{2}(E_2+E_1) + \frac{\hbar^2 k^2}{2m} + \frac{E_g}{2}\sqrt{\quad} = -\frac{E_g}{2} + \frac{\hbar^2 k^2}{2m} + \frac{E_g}{2}\sqrt{\quad} \approx \frac{\hbar^2 k^2}{2m} + \frac{\hbar}{m}(\vec{k}\cdot\vec{q})$ (1)

* The result above is consistent with $\left[\left\{\frac{1}{2m}P\cdot\alpha\cdot P + \mu_0 S g^* H + V(r)\right\}\frac{E_g}{E+E_g} - E\right]|\psi\rangle = 0$

Take $\vec{H}=0 \to E = -\frac{E_g}{2} + \frac{E_g}{2}\sqrt{1+\frac{4(\vec{k}\cdot\vec{\alpha}')^2}{E_g^2}}$, $\alpha' = \alpha/2m$

$\to E \approx \vec{k}\cdot\vec{\alpha}'$ this is (1) if we take $\alpha' \equiv \left(\frac{\hbar^2\vec{k}}{2m} + \frac{\hbar}{m}\vec{q}\right) \to$ effective mass.

(b) See Landau, Lifshitz, Q.M. Non-relativistic (V.3). Use k·P Hamiltonian in (a) an recognize Harmonic oscil.

$E = \left(n+\frac{1}{2}\right)\hbar\omega_c + \frac{p_z^2}{2m} - \mu_0 g H$; $\omega_c = \frac{|e|\alpha_z H}{mc}$, $\vec{H} = (0,0,H)$, $p_z = \hbar k_z \alpha_\ell$, m: free mass.

(c) Use semiclassical : $H = \frac{p^2}{2m} = E(1-E/E_g)$; $v = \frac{1}{\hbar}\frac{\partial E}{\partial k} = \frac{\hbar k}{m}\frac{1}{1+2E_F/E_g}$ $\Rightarrow$

$\Rightarrow \frac{d\vec{k}}{dt} = \frac{e}{c}\vec{v}\times\vec{B} \Rightarrow \hbar\frac{dk}{dt} = \frac{e}{c}\frac{\hbar k}{m}\frac{B}{1+2E_F/E_g} \Rightarrow \oint\frac{dk}{k} = \oint\frac{eB}{mc}\frac{dt}{1+2E_F/E_g} \Rightarrow 2\pi = \frac{eB}{mc}\frac{1}{1+2E_F/E_g}T \Rightarrow$

$\Rightarrow \omega_c = \frac{eB}{m_c^* c} \Rightarrow \boxed{m_c^* = m\,(1+2E_F/E_g)}$ ADD spin $E_0 \pm \mu g H = E^\pm\left(1+\frac{E^\pm}{E_g}\right)$; $E_0 \equiv E_F\left(1+\frac{E_F}{E_g}\right)$

E, ΔE, E^+, E^- $\quad \Delta E = \mu g^* H$

Solving for $\Delta E \to g^* = \frac{\sqrt{1-4E_0E_g+\mu gHE_g}}{\mu H}$

5c alternative)

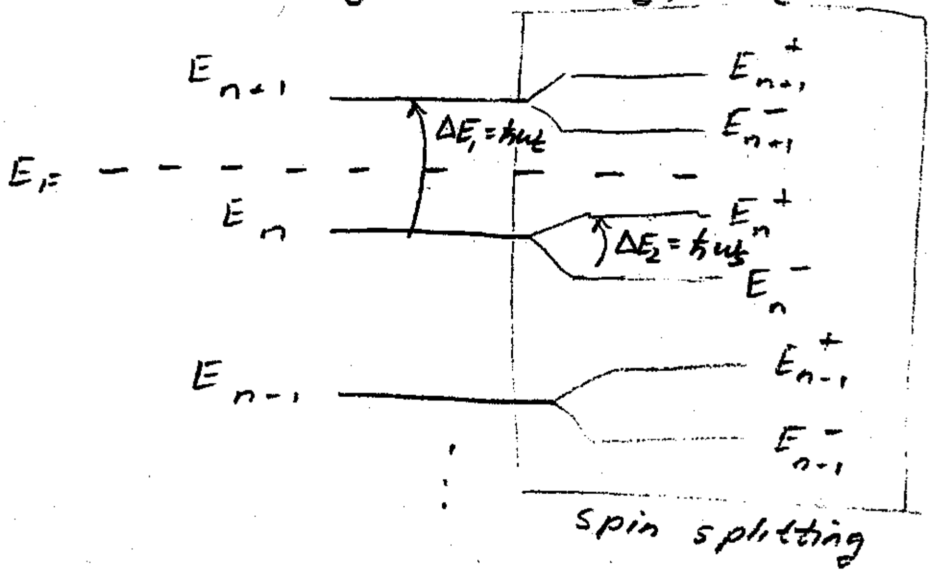
Rewriting the Schrodinger Equation as follows:

$$\underbrace{\left[\tfrac{1}{2m}\, p \cdot \alpha \cdot p \;+ \mu_B S \cdot g \cdot H \;+ V(r) \right]}_{\text{Harmonic Oscillator}} f(r) = \underbrace{E\left(\frac{E+E_g}{E_g}\right)}_{\text{adjusted energy}} f(r)$$

We recognize the Harmonic oscillator Hamiltonian with an adjusted energy: $E\left(\frac{E+E_g}{E_g}\right) = E(1+E/E_g)$

$\longrightarrow$ Eigenenergies are

$$E(1+E/E_g) = (n+\tfrac{1}{2})\hbar\omega_c \pm \mu_0 g H_z + P_z^2/2m$$



$$\Delta\left[E(1+E/E_g)\right] = \Delta E\,(1+E/E_g) + E\,(\Delta E/E_g)$$

$$= \Delta E + 2E\,\Delta E/E_g$$

$$= \Delta E\,(1 + 2E/E_g)$$

The excited states are near $E = E_F \longrightarrow$

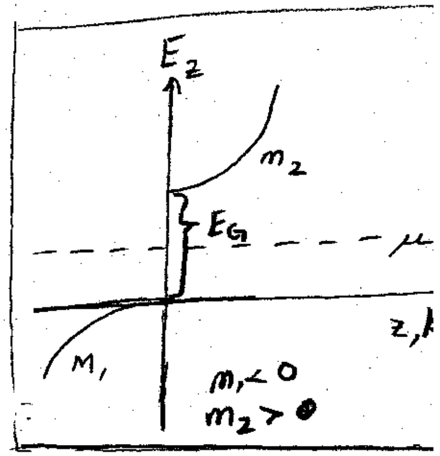
$$\left.\begin{matrix} \hbar \frac{eB}{m_0 c} = \hbar\omega_c \\ \mu g H = \hbar\omega_s \end{matrix}\right\} \cong \begin{Bmatrix} \Delta E_1 \\ \Delta E_2 \end{Bmatrix} (1 + 2E_F/E_g) = \begin{cases} \hbar \frac{eB}{m^* c}(1 + 2E_F/E_g), \\ \mu g^* H(1 + 2E_F/E_G) \end{cases}$$

$$\longrightarrow m^* = m_0\left(1 + \frac{2E_F}{E_g}\right) \qquad g^* = g_0 / (1 + 2E_F/E_g)$$

5a.) The Schrodinger Eq. is

$$-\frac{\hbar^2}{2m_*}\nabla^2 f(x,y,z) = E_{TOT}\, f(x,y,z)$$

We know $f(x,y,z)$ can be separated so that



$$f(x,y,z) = X(x)Y(y)Z(z)$$

and the energy splits up as well:

$$E_{TOT} = E_x + E_y + E_z .$$

In the xy-plane, we have free particles:

$$X(x) \sim e^{ik_x x} \quad , \quad k_x^2 = \frac{2mEx}{\hbar^2}$$

$$Y(y) \sim e^{ik_y y} \quad , \quad k_y^2 = \frac{2mEy}{\hbar^2}$$

Now we concentrate on the z-direction:

$$-\hbar^2 \frac{\partial^2 Z}{\partial z^2} = \begin{cases} -\frac{2m_1}{\hbar^2} E_z Z & , z<0 \\ -\frac{2m_2}{\hbar^2}(E_z - E_G)Z & , z>0 \end{cases} \qquad (1)$$

Continuity of the wave-function implies

$$\boxed{Z(0^+) = Z(0^-)} \qquad (2)$$

Integrating the Schrödinger eq. across $z=0$ gives another boundary condition:

$$\int_{0^-}^{0^+} -\frac{\hbar^2}{2m_*}\frac{\partial^2 Z}{\partial z^2}\, dz = 0$$

|| the R.H.S. is 0 because of (2)

$$\longrightarrow \quad \boxed{\frac{Z'(0^+)}{m_2} = \frac{Z'(0^-)}{m_1}} \qquad (3)$$

b.) We rewrite (1) to get

$$\frac{\partial^2 Z}{\partial z^2} = \begin{cases} \underbrace{-2m_1 E_z/\hbar^2}_{(+)\text{ since } m_1<0} \; Z & , z<0 \\ \underbrace{+2m_2(E_G - E_z)/\hbar^2}_{(+)} \; Z & , z>0 \end{cases} \qquad (4)$$

· 5b cont.)

Since $Z(\pm\infty) = 0$, (4) $\longrightarrow$

$$Z = \begin{cases} A\, e^{k_- z}, & z < 0 \ni k_-^2 = -\frac{2m_1 E_z}{\hbar^2} \\ B\, e^{-k_+ z}, & z > 0 \ni k_+^2 = \frac{2m_2 (E_G - E_z)}{\hbar^2} \end{cases} \qquad (5)$$

Continuity @ $z = 0 \longrightarrow A = B$. (3) $\longrightarrow$

$$-\frac{k_+}{m_2} = \frac{k_-}{m_1} \qquad (6)$$

Square both sides and use (5) to get

$$\frac{E_G - E_z}{m_2} = \frac{E_z}{m_1} \Longrightarrow \boxed{E_z = E_G\left(\frac{m_1}{m_1 - m_2}\right)} \qquad (7)$$

Thus, $\boxed{f(x,y,z) = A\, e^{i \vec{k}_{\parallel} \cdot \vec{r}}\, e^{\mp k_\pm z_\pm}} \qquad (8)$

$k_\parallel \equiv (k_x^2 + k_y^2)^{1/2}$

$\vec{r} \equiv \vec{x} + \vec{y}$

(c.) To find the dispersion relation, we use energy conservation on both sides of the junction:

<u>Left</u> $\qquad\qquad$ RIGHT

(9) $E = \frac{\hbar^2}{2m_1} k_\parallel^2 + \frac{\hbar^2}{2m_1} k_-^2 \qquad E = E_G + \frac{\hbar^2}{2m_2} k_\parallel^2 + \frac{\hbar^2}{2m_2} k_+^2$ (10)

$$\boxed{-\frac{k_+}{m_2} \overset{(6)}{=} \frac{k_-}{m_1}}$$

$$\frac{\hbar^2}{2}\left(\frac{k_\parallel^2}{m_1} + m_1 \frac{k_+^2}{m_2^2}\right) = E_G + \left(\frac{k_\parallel^2}{m_2} + \frac{k_+^2}{m_2}\right)\hbar^2/2$$

$$\frac{\hbar^2 k_\parallel^2}{2}\underbrace{\left(\frac{1}{m_1} - \frac{1}{m_2}\right)}_{\frac{m_2 - m_1}{m_1 m_2}} - E_G = \frac{\hbar^2 k_+^2}{2m_2}\underbrace{\left(1 - \frac{m_1}{m_2}\right)}_{\frac{m_2 - m_1}{m_2}}$$

$$\longrightarrow \frac{\hbar^2 k_\parallel^2}{2m_1} - \frac{E_G m_2}{m_2 - m_1} = \frac{\hbar^2 k_+^2}{2m_2} \qquad (11)$$

Thus, combining (10) & (11) $\longrightarrow$ $\boxed{E = \frac{-m_1 E_G}{[\text{illegible}]} + \frac{\hbar^2 k_\parallel^2}{2}\underbrace{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}_{\equiv 1/m^*}}$ (12)

d.) For a 2D e^--gas, the density of states is a constant:

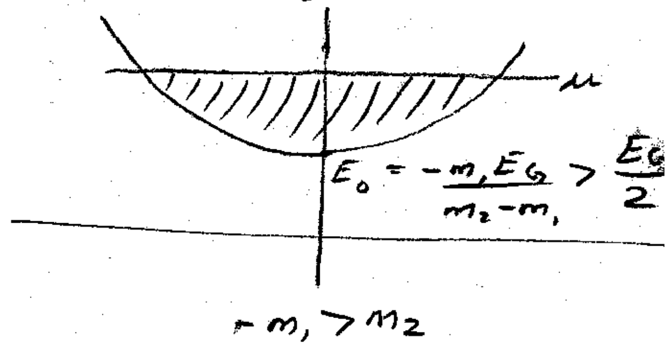
$$\mathcal{D}_{2D} = m^*/\hbar^2\pi$$

$$\mathcal{D}_{2D} = \frac{m_1 m_2}{m_1 + m_2}\frac{1}{\hbar^2\pi}$$

For $-m_1 > m_2$ (remember that m_1 is negative),

$$n = (\mu - E_0)\mathcal{D}(E)$$

$$= (\mu - E_0)\frac{m^*}{\pi\hbar^2}$$



For $-m_1 = m_2$, $\mathcal{D}_{2D} \to \infty$, so $n \to \infty$. (12) $\to$ NO DISPERSION

For $-m_1 < m_2$ $(\mu < E_0)$

$$p = \frac{(\mu - E_0)}{\pi\hbar^2}\frac{m_1 m_2}{m_1 + m_2}$$

density of holes