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Physics 732 **HOMEWORK ASSIGNMENT #1** **Spring 2010**

Deadline: March 30, 2010

1. a) A&M 33.4: a) :
Confirm the normalization $|\mathbf{R}\rangle = (2S)^{1/2}\, \mathbf{S}(\mathbf{R})\, |0\rangle$ and $\mathbf{S}(\mathbf{R'})\, \mathbf{S}_{+}(\mathbf{R})\, |\mathbf{R}\rangle = 2S|\mathbf{R'}\rangle$

b) Suppose (for S= ½) that we write $\langle \mathbf{k}|\, \mathbf{S}_{\perp}(\mathbf{R})\, \mathbf{S}_{\perp}(\mathbf{R}+a\hat{\mathbf{y}})|\mathbf{k}\rangle = (1/4)\cos\theta$. Referring to the relationship for $\langle \mathbf{k}|\, \mathbf{S}_{\perp}(\mathbf{R})\, \mathbf{S}_{\perp}(\mathbf{R'})|\mathbf{k}\rangle$ derived in class, what is the value of θ, the angle between spins on neighboring sites in the $\hat{\mathbf{y}}$ direction?

c) A&M 33.4: c) Show that $\langle \mathbf{k}|\, \mathbf{S}_{\perp}(\mathbf{R})|\mathbf{k}\rangle = 0$, which means that the phase of the spin wave is unspecified in state $|\mathbf{k}\rangle$.

2. Refer to pp. 661-663 of A&M; the spin susceptibility of a conduction electron gas at T = 0 K may be discussed by another method. Let $n_{\pm} \equiv (n/2)(1 \mp \zeta)$ be the concentration of spin-up (-down) electrons, i.e. parallel (antiparallel) to a magnetic field H.

a) Show that, in H, the total energy per volume in the spin-up band in a free-electron gas is

$E^{+} = E_0\, (1 - \zeta)^{5/3} + (n/2)\, \mu_B H\, (1 - \zeta)$,

where $E_0 = (3/10)n\varepsilon_F$ in terms of the Fermi energy ε_F in zero magnetic field. Find a similar expression for E^{-}.

b) Minimize the total energy per volume $E^{+} + E^{-}$ with respect to ζ and solve for the equilibrium value of ζ in the approximation $\zeta \ll 1$. Show then that the magnetization $M = (3/2)\, n\mu_B^2 H/\varepsilon_F$, as in the class discussion of Pauli paramagnetism.

c) We now consider the effect of exchange interactions among the conduction electrons. As a viable first approximation, we assume that electrons with parallel spins interact with each other with energy -V (with V > 0), while electrons with antiparallel spins do not interact with each other. Show that the additional term $-\,(1/8)\, V n^2\, (1 - \zeta)^2$ is added to E^{+} and find a similar expression for E^{-}.

d) Minimize the total energy and solve for ζ again in the limit $\zeta \ll 1$. Show that the magnetization is

$$M = \frac{3n\mu_B^2}{2\varepsilon_F - \frac{3}{2}Vn} H$$

Notice that there is peculiar behavior for $V > 4\varepsilon_F/3n$. One can easily show that at H=0 the total energy for the paramagnetic state with $\zeta = 0$ is unstable relative to a ferromagnetic state with finite ζ. This is called the Stoner criterion for ferromagnetism. (adapted from Kittel, ISSP) This kind of phase transition at T=0 is now glamorized with the label "quantum phase transition."

3. Consider the Landau theory of phase transitions at a tricritical point, for which

$F = g_0 + (1/2)\, a\, (T-T_0)\, P^2 + (1/6)\, g_6\, P^6$

a) Show that $\beta = ¼$, i.e. that $P \propto (T_0 - T)^{1/4}$ near the transition.

b) Show that the susceptibility $X = P/\ E)E{=}0 \propto |T_0 - T|^{-1}$ i.e. g = 1 as for the critical transition, but that the critical amplitude ratio is 4 rather than 2.

4. A&M 34–2.

2. *The London Equation for a Superconducting Slab*

Consider an infinite superconducting slab bounded by two parallel planes perpendicular to the y-axis at $y = \pm d$. Let a uniform magnetic field of strength H_0 be applied along the z-axis.

(a) Taking as a boundary condition that the parallel component of **B** be continuous at the surface, deduce from the London equation (34.7) and the Maxwell equation (34.6) that within the superconductor

$$\mathbf{B} = B(y)\hat{\mathbf{z}}, \quad B(y) = H_0 \frac{\cosh(y/\Lambda)}{\cosh(d/\Lambda)}. \qquad \textbf{(34.42)}$$

(b) Show that the diamagnetic current density flowing in equilibrium is

$$\mathbf{j} = j(y)\hat{\mathbf{x}}, \quad j(y) = \frac{c}{4\pi\Lambda} H_0 \frac{\sinh(y/\Lambda)}{\cosh(d/\Lambda)}.$$

(c) The magnetization density at a point within the slab is $\mathbf{M}(y) = (\mathbf{B}(y) - \mathbf{H}_0)/4\pi$. Show that the average magnetization density (averaged over the thickness of the slab) is

$$\overline{M} = -\frac{H_0}{4\pi}\left(1 - \frac{\Lambda}{d}\tanh\frac{d}{\Lambda}\right), \qquad \textbf{(34.43)}$$

and give the limiting form for the susceptibility when the slab is thick ($d \gg \Lambda$) and thin ($d \ll \Lambda$).

5. A&M 34–4. [Note that you do not need to derive eq. (34.46).]

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4. *The Cooper Problem*

Consider a pair of electrons in a singlet state, described by the symmetric spatial wave function

$$\phi(\mathbf{r} - \mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3} \chi(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{r}-\mathbf{r}')}. \tag{34.45}$$

In the momentum representation the Schrödinger equation has the form

$$\left(E - 2\frac{\hbar^2 k^2}{2m}\right)\chi(\mathbf{k}) = \int \frac{d\mathbf{k}'}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}')\chi(\mathbf{k}'). \tag{34.46}$$

We assume that the two electrons interact in the presence of a degenerate free electron gas, whose existence is felt only via the exclusion principle: Electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint:

$$\chi(\mathbf{k}) = 0, \quad k < k_F. \tag{34.47}$$

We take the interaction of the pair to have the simple attractive form (cf. Eq. (34.16)):

$$\begin{aligned} V(\mathbf{k}_1, \mathbf{k}_2) &= -V, \qquad \mathcal{E}_F \leqslant \frac{\hbar^2 k_i^2}{2m} \leqslant \mathcal{E}_F + \hbar\omega, \quad i = 1, 2; \\ &= 0, \qquad \text{otherwise,} \end{aligned} \tag{34.48}$$

and look for a bound-state solution to the Schrödinger equation (34.46) consistent with the constraint (34.47). Since we are considering only one-electron levels which in the absence of the attraction have energies in excess of $2\mathcal{E}_F$, a bound state will be one with energy E less than $2\mathcal{E}_F$, and the binding energy will be

$$\Delta = 2\mathcal{E}_F - E. \tag{34.49}$$

(a) Show that a bound state of energy E exists provided that

$$1 = V \int_{\mathcal{E}_F}^{\mathcal{E}_F + \hbar\omega} \frac{N(\mathcal{E})\, d\mathcal{E}}{2\mathcal{E} - E}, \tag{34.50}$$

where $N(\mathcal{E})$ is the density of one-electron levels of a given spin.

(b) Show that Eq. (34.50) has a solution with $E < 2\mathcal{E}_F$ for arbitrarily weak V, provided that $N(\mathcal{E}_F) \neq 0$. (Note the crucial role played by the exclusion principle: If the lower cutoff were not $\mathcal{E}_F$, but 0, then since $N(0) = 0$, there would *not* be a solution for arbitrarily weak coupling).

(c) Assuming that $N(\mathcal{E})$ differs negligibly from $N(\mathcal{E}_F)$ in the range $\mathcal{E}_F < \mathcal{E} < \mathcal{E}_F + \hbar\omega$, show that the binding energy is given by

$$\Delta = 2\hbar\omega \frac{e^{-2/N(\mathcal{E}_F)V}}{1 - e^{-2/N(\mathcal{E}_F)V}}, \tag{34.51}$$

or, in the weak-coupling limit:

$$\Delta = 2\hbar\omega e^{-2/N(\mathcal{E}_F)V}. \tag{34.52}$$