Physics 721: Homework # 2
Due Wednesday October 7

1. Dressed states and adiabatic passage

Consider a two-level system subject to an E/M field of fixed intensity (Rabi frequency) but varying detuning $\delta(t)$ (Figure 1). Assume that initially the detuning is large ($|\delta(0)| \gg |\Omega|$) and negative ($\nu < \omega$). The detuning is then slowly scanned through resonance.

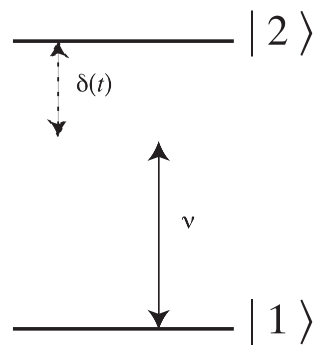

Figure 1: Problem 1

- Determine the instantaneous (adiabatic) eigenenergies and eigenstates as a function of $\delta(t)$.
- If the initial state of the system is $|\psi\rangle = |1\rangle$, what is the final state, including relevant phases? Consider now the situation when the detuning is adiabatically returned to its initial value. What is the final state vector of the system, including relevant phases? Provide physical interpretation for various accumulated phases.
- For a given Ω, how slow should the detuning change such that the resulting evolution is adiabatic?

2. Raman transitions and adiabatic elimination

Consider a 3-level system interaction with two separate E/M fields, as shown in Figure 2. Assume the fields only interact with atoms via the couplings shown (e.g. field 1 does not couple $|2\rangle$ to $|3\rangle$).

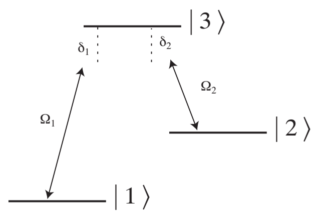

Figure 2: Problem 2

- Derive equations of motion for slowly varying probability amplitudes in the rotating frame $(c_1(t), c_2(t), c_3(t))$.
- Assuming that the detunings $\delta_1 = \nu_1 - \omega_{13}$ and $\delta_2 = \nu_2 - \omega_{23}$ are large but similar (e.g. $\delta_1 \simeq \delta_2$ and $\delta_1, \delta_2 \gg \Omega_1, \Omega_2$), show that the dynamics of the system can be described by an effective Hamiltonian,

  $$\hat{H}_{eff} = \hbar\tilde{\omega}_1|1\rangle\langle 1| + \hbar\tilde{\omega}_2|2\rangle\langle 2| + \hbar\Omega_{eff}|1\rangle\langle 2| + \hbar\Omega^*_{eff}|2\rangle\langle 1|$$

  and derive $\tilde{\omega}_{1,2}$ and Ω_{eff}.

  [Hint: for large detunings you can "adiabatically eliminate" the excited state amplitudes, i.e. set $\dot{c}_3(t) = 0$ and express c_3 in terms of $c_{1,2}$. The remaining evolution equations can then be described in terms of the above effective Hamiltonian.]

3. Coherent control of spontaneous emission

Consider the transition from a bound state ($|b\rangle$ with energy $\epsilon_0 = 0$) into a 1-D continuum induced by a quasi-monochromatic field with a time-dependent, slowly varying amplitude $f(t)$. The generic Hamiltonian for this system is given by

$$\hat{H} = \sum_{k=0}^{\infty} (\hbar\omega_k - \hbar\nu)|k\rangle\langle k| - \hbar f(t) \sum_k g_k |b\rangle\langle k| + h.c.$$

The continuum states $|k\rangle$ represent 1-d momentum states. Assume that $\nu \gg \omega_{k=0}$, for all relevant k, $g_k = g$ is constant, and that the density of momentum states near ν is ρ_0. (The simplified dispersion relation $\omega_k = v_0|k|$ can be used, if needed). Assume that the bound state corresponds to a localized state at $r = 0$, and that only positive values of r are allowed. The latter condition implies that without coupling to the bound state any incoming wave-packet would simply reflect from the $r = 0$ surface.

This model represents a 1-dimensional "toy model" for laser-induced photoionization or photodissociation. This model also describes the coupling of a single mode cavity to free space modes with one semi-transparent mirror (the time dependent coupling represents modulation of the mirror reflectivity). $r = 0$ corresponds to the positions of the "ion", center of mass of "molecule" or the mirror surface, respectively.

- Assuming that the system is in the bound state $|b\rangle$ at $t = 0$ derive an expression for probability amplitude $c_b(t)$ as a function of $f(t)$.

- Derive an expression for the real-space wavefunction $c(r,t) = \int_{-\infty}^{\infty} dk \ c_k(t) e^{ikr}$ of the outgoing (continuum-state) wavepacket at $r = 0$.

- Consider now an inverse problem (such as photoassociation). Suppose that the incoming wavepacket $c^{\text{in}}(0,t)$ is given. Assume that it is far from the center ($r = 0$) at some initial time t_i, when the bound state is empty. Derive an expression for the bound state amplitude $c_b(t)$ and in terms of $c^{\text{in}}(0,t)$ and $f(t)$.

  [Hint: Equations for c_k should now be solved with initial conditions $c_k(t_i)$.]

- *Extra Point* For the given shape of the incoming wavepacket, is it possible to find a function $f(t)$ such that at $t \to \infty$ $c_b \to 1$, i.e. the system is in the bound state with unity probability? If so, derive the condition (such as differential equation) for $f(t)$ as a function of incoming wavepacket. Provide physical interpretation for your results.

  [Hint: The best way to approach this problem is determine the conditions for which the amplitude of the outgoing wavepacket and its derivative are zero at all time. Alternatively, it may be possible to use some of the previous results together with the fact the dynamics should be time reversible.]

  N.B. This is the basic idea behind what is called "coherent control". In principle, it be applied, for example, to the problem of efficient and coherent photo-association of BECs or to the problem of coherent conversion between photonic and atomic qubits. These could be potential topics for the term project.