**Physics 721: Homework # 1**
**Due Wednesday September 23**
website: http://www.physics.umd.edu/courses/Phys721

**1. Rabi oscillations**

Consider a two-level atom with states $|g\rangle, |e\rangle$ separated by an energy $\hbar\omega$:

$$H = \hbar\omega|e\rangle\langle e| - 2\hbar\Omega\cos(\nu t + \phi)|e\rangle\langle g| + \text{H.c.} \tag{1}$$

- Transform $H$ to a rotating frame by application of an appropriate (time-dependent) unitary transformation. Rewrite this new Hamiltonian using Pauli matrices defined by $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, $\sigma_x = |e\rangle\langle g| + \text{H.c.}$, $\sigma_y = I\sigma_x\sigma_z$.

- Make the rotating wave approximation, neglecting terms in the new rotating frame oscillating at the frequency $\pm|\omega+\nu|$. Making the identification $\delta = \omega - \nu$, evaluate the probability that an initial state $|g\rangle$ makes the transition to the state $|e\rangle$ after a time $t$ as a function of $\delta$. For simplicity, you can use units of frequency such that $\Omega = 1$. What happens for $\delta = 0$? Find the maximum probability for a given $\delta$.

- What is the quantum state of the system, starting in an initial state $|g\rangle$, for $\delta = 0$, as a function of $\phi$ and $t$? What is the expectation value of $\sigma_x$ and $\sigma_y$ as a function of $\phi$ and $t$?

- In class we noted that the *formal* solution for the propagator is given by the series

$$U(t) = 1 - \frac{i}{\hbar}\int_0^t H(t')dt' - \frac{1}{\hbar^2}\int_0^t\int_0^{t'} H(t')H(t'')dt''dt' + \ldots \tag{2}$$

  For periodic, time-dependent Hamiltonians with period $\tau$, show that $U(t+n\tau) = U(t)U(\tau)^n$. This may be easiest by using the time-ordering operator and writing $U(\tau) = \mathcal{T}\exp(-\frac{i}{\hbar}\int_0^\tau H(t)dt)$.

- Finding $U(\tau)$ can tell us about the dynamics at long times for this situation. Towards this end, we can take advantage of the following property of unitary operators: all unitary operators can be written as $\exp(-iA)$, where $A$ is a hermitian operator.[1]

  Make the identification $U(\tau) = \exp(-\frac{i\tau}{\hbar}(H_0 + H_1 + \ldots))$ where $H_i$ are Hermitian operators, to be determined. Expanding $U(\tau)$ for small $\tau$, identifying $H_0 = \frac{1}{\tau}\int_0^\tau H(t)dt$ as the *average* Hamiltonian over one period, and comparing to Eqn. (2), find the first correction $H_1$.

- Going now back to the original time-dependent Hamiltonian, for $\delta = 0$ but *not* making the rotating wave approximation, find $H_0$ and $H_1$. What is the size of the first "correction" term?

[1] The proof: all unitary operators are diagonalizable and have eigenvalues of the form $\exp(-i\phi_j)$ with the phases $\phi_j$ real. Let $V$ be the matrix which diagonalizes $U$ (i.e., $VUV^\dagger$ is diagonal). We define $\tilde{A}_{ij} = \phi_j\delta_{ij}$ where $\delta_{ij}$ is the Kroenecker delta (zero if $i \neq j$, 1 if $i = j$). Then $A = V^\dagger\tilde{A}V$ is the Hermitian operator with the property $U = \exp(-iA)$.

**2. Hyperfine interaction in an external magnetic field**

Consider the hyperfine interaction Hamiltonian as derived in class, with the addition of an external magnetic field

$$H = A\vec{I} \cdot \vec{J} + B(g_J \mu_0 J_z - g_I \mu_0 I_z) \tag{3}$$

where $\vec{J}$ and $\vec{I}$ are the combined orbital and spin angular momentum and the nuclear spin, respectively with $\hbar = 1$. $[J_\mu, J_\nu] = i\epsilon_{\mu\nu\tau} J_\tau$, and similarly for $\vec{I}$.

- Recall that the total angular momentum is the vector $\vec{F} = \vec{J} + \vec{I}$. Show that $F^2 = (F \cdot F)$ and $F_z$ are constants of the motion (i.e., commute with the Hamiltonian).
- Consider the scenario when $J = 1/2$ (as occurs, for example, in the ground state alkali atoms with term ${}^2S_{1/2}$). In this scenario, the Hamiltonian becomes solvable analytically. You can start by consider the effect of the Hamiltonian on the states $|m_I, 1/2\rangle$ and $|m_I + 1, -1/2\rangle$, where the $\pm 1/2$ label the $J_z$ eigenvalues of the state and $m_I$ is the $I_z$ eigenvalue. Write down the Hamiltonian in the subspace with total $m_F = m_I + 1/2$. This should be a 2x2 matrix.
- Find the eigenvalues and eigenvectors of this Hamiltonian.
- For the specific case of $I = 1$, sketch or plot the energies as a function of external magnetic field $B$ using appropriate scale-free parameters. You may set $g_J = 1, g_I = 0.1$ for simplicity. This should include the solutions for all allowed $F^2$ and $m_F$ values. Consider the range $B = 0$ to $B \gg A/|(g_J\mu_0 + g_I\mu_0)|$. Label the $F^2$ and $m_F$ values for each line at low and high field.
- Often systems become sensitive to experimental errors. One measure of this sensitivity is the dependence of the relative energies between two states to small variations of external parameters. To illustrate this, evaluate the energy difference between pairs of states differing in $m_F$ by 1. For a given pair, is there a value of $B$ such that $d/dB(E_{\text{state1}} - E_{\text{state2}}) = 0$? You need only find one such pair.

## 3. Optical transitions and angular momentum

An atom of total angular momentum $F$ has a spontaneous radiation rate $A$. It radiates to a lower level with angular momentum $F' = F - 1$. The problem is to find the rates for the various allowed transitions, i.e. the fraction of the radiation that goes into each of the possible transitions $(F, m) \rightarrow (F', m')$. Each of the rates is proportional to $|\langle F, m_F | Y_{1,q} | F', m'_F \rangle|^2$ with $q = \pm 1, 0$. The rates can be found by either direction evaluation of matrix elements or by applying the following considerations:

(1) The sum of the rates out of each state $F, m$ must equal $A$

(2) The sum of the rates into each state $F', m'$ must equal $A\frac{2F+1}{2F'+1}$.

(3) An unpolarized mixture of radiators in level $F$ must emit equal intensities of light with each of the three polarization components $(z, \sigma_\pm)$.

(4) The rate for a transition $(F, m \rightarrow F', m')$ must be the same as for $(F, -m \rightarrow F', -m')$.

Consider the situation $F = 2, F' = 1$. Designation the transitions by letters as follows:

$$\begin{array}{llll} a: & m = 2 & \rightarrow & m' = 1 \\ b: & m = 1 & \rightarrow & m' = 1 \\ c: & m = 0 & \rightarrow & m' = 1 \\ d: & m = 1 & \rightarrow & m' = 0 \\ e: & m = 0 & \rightarrow & m' = 0 \end{array}$$

- The Wigner-Eckart theorem (see e.g. Sakurai) can be used to evaluate matrix elements in terms of an $m$-independent quantity. Note however that $F$ involves orbital, electron spin and nuclear spin components. Does the W.-E. theorem as stated (i.e. in Sakurai) still apply? Explain why.

- Find the rates for $a$ through $e$ in terms of $A$ using the appropriate version of the Wigner-Eckart theorem and make a figure of your results. (Clebsch-Gordan coefficients can either by worked out from first principles, taken from a table in a quantum mechanics or spectroscopy text or computed with Mathematica)

- Using the symmetry considerations and conservation of probabilities (i.e. the total number of decaying atoms from level $F$ should be equal to the total number arriving to $F'$) show rules (2) and (4) must be true.

- Find the rates for $a$ through $e$ using rules (1)–(4), and make a figure of your results.

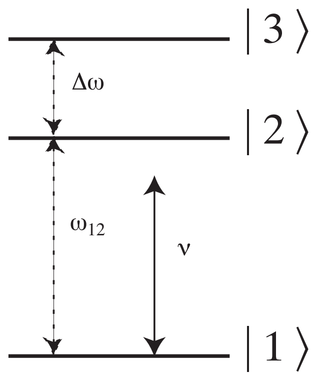


Figure 1: Problem 4

**4. Two-level approximation**

Consider a three-level system, shown in Figure 1, with two excited states, each coupled to the ground state with identical selection rules and matrix elements. The energy spacing between states $|2\rangle$ and $|3\rangle$ is $\Delta\omega$, which is much smaller than the optical frequency $\nu$ and the frequency separation between 1 and 2, $\omega_{12}$. An oscillating (optical) field is tuned to resonance for the transition $|1\rangle \rightarrow |2\rangle$. It will, in principle, also induce transitions from $|1\rangle$ to $|3\rangle$.

- Derive equations of motion for the probability amplitudes $c_i$ associated with each level, i.e., by writing the state of the system as $|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$. Assume that the applied field has frequency $\nu$ and equal Rabi frequencies for both of the optical transitions ($\Omega$).

- Suppose now that $\nu = \omega_{12}$ and $\Delta\omega$ is much larger than the Rabi frequency $\Omega$. Evaluate the effect of state $|3\rangle$ on Rabi oscillations to the second order in $1/\Delta\omega$.

- Show that the leading order correction to the Rabi dynamics due to the state $|3\rangle$ can be compensated by slightly detuning the laser field. Determine the detuning $\delta = \nu - \omega_{12}$ needed. Provide physical explanation for your result. Describe the physics behind the next relevant correction.

- Under which conditions (i.e. Rabi frequency and interaction time) can the effect of state $|3\rangle$ be neglected? In the case involving $D_1$ transition line in $^{87}$Rb the hyperfine splitting of the excited state is about 0.8 GHz. Can we treat a transition between hyperfine sublevels as a two-level system for laser-cooled or room temperature atoms? The natural linewidth of the excited state of $^{87}$Rb is about 5 MHz and the Doppler broadening at room temperature is about 500 MHz.