8) $H_0 = \hbar\omega\, b^\dagger b + \frac{1}{2}\hbar\omega\, a^\dagger a$

$H_I = V_0\left(a^\dagger a^\dagger b + a a b^\dagger\right)$

a) $i\hbar\dot{a} = [a, H] \qquad H = H_0 + H_I$

$$= a\left\{\hbar\omega\left(b^\dagger b + \tfrac{1}{2}a^\dagger a\right) + V_0\left(a^\dagger a^\dagger b + aab^\dagger\right)\right\}$$
$$-\left\{\hbar\omega\left(b^\dagger b + \tfrac{1}{2}a^\dagger a\right) + V_0\left(a^\dagger a^\dagger b + aab^\dagger\right)\right\}a$$
$$= \tfrac{1}{2}\hbar\omega\left(aa^\dagger a - a^\dagger aa\right) + V_0\left(aa^\dagger a^\dagger - a^\dagger a^\dagger a\right)b$$
$$= \tfrac{1}{2}\hbar\omega\,[a, a^\dagger]a + V_0\left[(1 + a^\dagger a)a^\dagger - a^\dagger(aa^\dagger - 1)\right]b$$

so
$$\boxed{\begin{aligned}\dot{a} &= -\frac{i\omega}{2}a - \frac{i2V_0}{\hbar}a^\dagger b\\ \dot{a}^\dagger &= \frac{i\omega}{2}a^\dagger + \frac{i2V_0}{\hbar}ab^\dagger\end{aligned}}$$

These give
$$a(t) = a_0\left[\cosh\frac{\Omega t}{2\hbar} - i\frac{\hbar\omega}{\Omega}\sinh\frac{\Omega t}{2\hbar}\right] + a_0^\dagger\left[-\frac{i4V_0\beta}{\Omega}\sinh\frac{\Omega t}{2\hbar}\right]$$
$$a^\dagger(t) = a_0\left[\frac{i4V_0\beta}{\Omega}\sinh\frac{\Omega t}{2\hbar}\right] + a_0^\dagger\left[\cosh\frac{\Omega t}{2\hbar} + i\frac{\hbar\omega}{\Omega}\sinh\frac{\Omega t}{2\hbar}\right]$$

where $\Omega = \sqrt{16V_0^2|\beta|^2 - \hbar^2\omega^2}$

However, this solution gets ugly.

Instead, do the above for $H = H_I$ only

Then
$$\boxed{\begin{aligned}a(t) &= a_0\cosh\Omega t - ia_0^\dagger e^{-i\phi}\sinh\Omega t\\ a^\dagger(t) &= a_0^\dagger\cosh\Omega t + ia_0 e^{i\phi}\sinh\Omega t\\ &\text{where } \Omega = \frac{2V_0}{\hbar}|\beta|\end{aligned}}$$

8) b)

$\langle n \rangle = \langle 0 | a^\dagger a | 0 \rangle$

Only terms of $a_0 a_0^\dagger$ connect $|0\rangle$ to $|0\rangle$

$= \langle 0 | a_0 a_0^\dagger \sinh^2 \Omega t | 0 \rangle$

$$\boxed{\langle n \rangle = \sinh^2 \Omega t}$$

$\langle x \rangle = \frac{1}{2} \langle 0 | a + a^\dagger | 0 \rangle = 0$

$\langle y \rangle = \frac{1}{2i} \langle 0 | a - a^\dagger | 0 \rangle = 0$

$\langle x^2 \rangle = \frac{1}{4} \langle 0 | a^\dagger a^\dagger + 2a^\dagger a + aa + 1 | 0 \rangle$

$= \frac{1}{4} \langle a_0 a_0^\dagger \rangle \left[\cosh^2 \Omega t + \sinh^2 \Omega t + i \sinh \Omega t \cosh \Omega t (e^{i\phi} + e^{-i\phi}) \right]$

$= \frac{1}{4} \left[\cosh(2\Omega t) - \sinh(2\Omega t) \cos\phi \right]$

$= \frac{1}{4} \left[e^{2\Omega t} \left(\frac{1}{2} - \frac{1}{2}\cos\phi \right) + e^{-2\Omega t} \left(\frac{1}{2} + \frac{1}{2}\cos\phi \right) \right]$

$\langle x^2 \rangle = \frac{1}{4} \left[e^{2\Omega t} \sin^2\left(\frac{\phi}{2}\right) + e^{-2\Omega t} \cos^2\left(\frac{\phi}{2}\right) \right] = \Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$

likewise

$\langle y^2 \rangle = \frac{1}{4} \left[e^{2\Omega t} \cos^2\left(\frac{\phi}{2}\right) + e^{-2\Omega t} \sin^2\left(\frac{\phi}{2}\right) \right] = \Delta y^2 = \langle y^2 \rangle - \langle y \rangle^2$

$\langle n^2 \rangle = \langle 0 | a^\dagger a a^\dagger a | 0 \rangle \longleftarrow$ only keep $a_0 a_0^\dagger a_0 a_0^\dagger$ and $a_0 a_0 a_0^\dagger a_0^\dagger$

after some work:

$= \cosh^2 \Omega t \sinh^2 \Omega t \underbrace{\langle 0 | a_0 a_0 a_0^\dagger a_0^\dagger | 0 \rangle}_{=2} + \sinh^4 \Omega t \underbrace{\langle 0 | a_0 a_0^\dagger a_0 a_0^\dagger | 0 \rangle}_{=1}$

$= \sinh^4 \Omega t + 2(1 + \sinh^2 \Omega t) \sinh^2 \Omega t$

$$\boxed{\langle n^2 \rangle = 3 \sinh^4(\Omega t) + 2 \sinh^2(\Omega t)}$$