HW 7

1) a) $\Delta = \frac{\Omega^2}{4\delta} \quad (\delta >> \Gamma, \Omega)$

From OBE: $U = \frac{\hbar\delta}{2}\ln\left(1 + \frac{\Omega^2/2}{\delta^2 + \Gamma^2/4}\right)$

For $\Omega << \Gamma$, $\quad U \approx \frac{\hbar\delta}{2} \cdot \frac{\frac{1}{2}\Omega^2}{\delta^2 + \frac{1}{4}\Gamma^2}$

Note: for $\delta >> \Gamma, \Omega \rightarrow U \approx \frac{\Omega^2\hbar}{4\delta} = \hbar\Delta$ from above

b) OBE $\rightarrow$ $U = \frac{\hbar\delta}{2}\ln\left(1 + \frac{\frac{1}{2}\Omega^2}{\delta^2 + \frac{1}{4}\Gamma^2}\right)$

Dressed $\rightarrow$ $\Delta = \frac{1}{2}\left[\sqrt{\delta^2 + \Omega^2} - \delta\right]$

Now take $\Omega >> \Gamma$ and $\delta \lesssim \Gamma$

For simplicity, let $\delta^2 + \Gamma^2/4 = \frac{4}{3}\delta^2 \quad (\delta^2 = \frac{3}{4}\Gamma^2)$

OBE: $U = \frac{\hbar\delta}{2}\ln\left(1 + \frac{\Omega^2/2}{\frac{4}{3}\delta^2}\right)$

$\sim \frac{\hbar\delta}{2}\left[\ln\left(\frac{\Omega^2}{\delta^2}\right) - \ln\left(\frac{8}{3}\right)\right]$

$\sim \hbar\delta\ln\left(\frac{\Omega}{\delta}\right)$

Dressed: $\Delta \sim \frac{1}{2}\Omega\left(1 + \frac{\delta^2}{2\Omega^2}\right) - \frac{1}{2}\delta$

$\sim \frac{1}{2}(\Omega - \delta)$

$\sim \frac{1}{2}\Omega$

$= \frac{\delta}{2}\frac{\Omega}{\delta}$

OBE: shift $\sim \delta\ln\left(\frac{\Omega}{\delta}\right)$

Dressed: shift $\sim \delta\frac{\Omega}{\delta}$

2A)

HW#7

From Lecture 12 we had dressed levels with eigenvalues $E = \pm \frac{1}{2}(\delta^2+\Omega^2)^{1/2}$ and eigenvectors

$$|1,N\rangle = \sin\theta\,|g,N+1\rangle + \cos\theta\,|e,N\rangle$$
$$|2,N\rangle = \cos\theta\,|g,N+1\rangle - \sin\theta\,|e,N\rangle$$

in that lecture you were asked to confirm the solution for these eigenvectors as

$$\sin 2\theta = \Omega/(\Omega^2+\delta^2)^{1/2} \quad \text{and} \quad \cos 2\theta = \delta/(\Omega^2+\delta^2)^{1/2}$$

this may be done by writing $\mathcal{H}\Psi = E\Psi$ for one of the eigenvectors (recall that

$$\begin{array}{c c} & \begin{array}{cc} e & g \end{array} \\ \mathcal{H}/\hbar = \begin{array}{c} e \\ g \end{array} & \begin{pmatrix} -\delta/2 & \Omega/2 \\ \Omega/2 & \delta/2 \end{pmatrix} \end{array}$$

so $\mathcal{H}\Psi = E\Psi$ gives:

$$-\delta\sin\theta + \Omega\cos\theta = (\delta^2+\Omega^2)\sin\theta$$
$$\Omega\sin\theta + \delta\cos\theta = (\delta^2+\Omega^2)\cos\theta$$

multiplying the two equations and simplifying yields:

$$2\delta^2\sin\theta\cos\theta = \delta\Omega(\cos^2\theta - \sin^2\theta) \quad \text{or}$$

$$\frac{\sin 2\theta}{\cos 2\theta} = \Omega/\delta$$ which is equivalent to the expression in the notes of Lecture #12, p.10.

Solutions 7

2.B $\Gamma_{j \to i} \propto |\langle j | \hat{d}\cdot\hat{E} | i \rangle|^2$ with the usual constants.

$$\langle 1 | \hat{d}\cdot\hat{E} | 1 \rangle = (\sin\theta \langle g, N+1| + \cos\theta \langle e, N|)\, \hat{d}\cdot\hat{E}\, (\sin\theta |g, N\rangle + \cos\theta |e, N-1\rangle)$$

$= \sin\theta \langle g, N+1 | \hat{d}\cdot\hat{E} | g, N\rangle \sin\theta +$ ①

$\cos\theta \langle e, N | \hat{d}\cdot\hat{E} | g, N\rangle \sin\theta +$ ②

$\sin\theta \langle g, N+1 | \hat{d}\cdot\hat{E} | e, N-1\rangle \cos\theta +$ ③

$\cos\theta \langle e, N | \hat{d}\cdot\hat{E} | e, N-1\rangle \cos\theta$ ④

$\hat{d}$ is off diagonal $|e\rangle\langle g|$, eliminating ① and ④

the ~~[illegible]~~ energy conservation eliminates ③ $|g\rangle \to |e\rangle$

so only ② contributes.

$$\Gamma_{1 \to 1} = \Gamma (\sin\theta \cos\theta)^2$$

Similarly, there will be a single term, the $|e, N\rangle \to |g, N\rangle$ term (note that N, the number of laser photons doesn't change in the part that contributes — the emitted photon is not part of the laser field) and we get

$$\Gamma_{2 \to 2} = \Gamma (\sin\theta \cos\theta)^2$$

$$\Gamma_{2 \to 1} = \Gamma \sin^4\theta$$

$$\Gamma_{1 \to 2} = \Gamma \cos^4\theta$$

$\Gamma_1 = \Gamma_{1\to1} + \Gamma_{1\to2} = \Gamma \cos^2\theta$

$\Gamma_2 = \Gamma_{2\to1} + \Gamma_{2\to2} = \Gamma \sin^2\theta$

i.e., the lifetime of a dressed level is $\Gamma \times$ its $|e\rangle$ fractional population.

Solutions

2c $$\dot{P}_1 = -P_1\Gamma_{1\to 2} + P_2\Gamma_{2\to 1}$$

$$\dot{P}_2 = -P_2\Gamma_{2\to 1} + P_1\Gamma_{1\to 2}$$

the contribution to, e.g. $\dot{P}_1$ from decays from $|1\rangle$ and to $|1\rangle$ cancel out: $-P_1\Gamma_{1\to 1} + P_1\Gamma_{1\to 1}$ (there are 4 terms affecting the population of each level - four arrows either starting or terminating on each level - but two cancel out).

set $\dot{P}_1 = \dot{P}_2 = 0$, solve for

$$P_2 = P_1\frac{\Gamma_{1\to 2}}{\Gamma_{2\to 1}}$$ along with $P_1 + P_2 = 1$

gives:

$$P_1 = \frac{\Gamma_{2\to 1}}{\Gamma_{2\to 1} + \Gamma_{1\to 2}} \qquad P_2 = \frac{\Gamma_{1\to 2}}{\Gamma_{2\to 1} + \Gamma_{1\to 2}}$$

and $$P_1 = \frac{\sin^4\theta}{\sin^4\theta + \cos^4\theta} \qquad P_2 = \frac{\cos^4\theta}{\sin^4\theta + \cos^4\theta}$$

Solution

2D.

$$F = -\nabla(\text{dressed potentials}) =$$

$$\mp \nabla \frac{\hbar}{2}\left(\Omega^2_{(x)} + \delta^2\right)^{1/2}$$

$$= \mp \frac{\hbar}{2} \cdot \frac{1}{2} \frac{1 \cdot 2\Omega \nabla\Omega}{(\Omega^2+\delta^2)^{1/2}}$$

$$= \mp \frac{\hbar}{2} \frac{\Omega \nabla\Omega}{(\Omega^2+\delta^2)^{1/2}}$$

$$F_{ave} = \frac{\hbar}{2} \frac{\Omega\nabla\Omega}{(\Omega^2+\delta^2)^{1/2}} \left[P_1 - P_2\right]$$

$$= -\frac{\hbar}{2} \frac{\Omega\nabla\Omega}{(\Omega^2+\delta^2)^{1/2}} \left\{ \frac{(\sin^4\theta - \cos^4\theta)}{(\sin^4\theta + \cos^4\theta)} \right.$$

where $\tan 2\theta = -\frac{\Omega}{\delta} = \frac{\sin 2\theta}{\cos 2\theta} = -\Omega/\delta$

$$\sin 2\theta = -\Omega/(\Omega^2+\delta^2)^{1/2}$$

$$\cos 2\theta = \delta/(\Omega^2+\delta^2)^{1/2}$$

$\sin^4\theta$ $\quad$ $\sin 2\theta = 2\sin\theta\cos\theta.$ $\quad$ $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos^4\theta$ $\quad$ $\cos 2\theta = \cos^2\theta - \sin^2\theta.$ $\quad$ $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$$\sin^4\theta = \sin^2\theta\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\frac{1}{2}(1-\cos 2\theta)$$

$$\cos^4\theta = \cos^2\cos^2 = \frac{1}{4}(1+\cos 2\theta)(1+\cos 2\theta)$$

$$\frac{\sin^4 - \cos^4\theta}{\sin^4 + \cos^4} = \frac{(\sin^2+\cos^2)(\sin^2-\cos^2)}{\sin^4+\cos^4} =$$

$$\frac{\sin^2 - \cos^2}{\frac{1}{4}(1-\cos 2\theta)^2 + \frac{1}{4}(1+\cos 2\theta)^2} = \frac{\cancel{2}\cos 2\theta}{\frac{1}{4}(1-2\cos 2\theta + \cos^2 2\theta) + \frac{1}{4}(1+2\cos 2\theta + \cos^2 2\theta)}$$

$$\frac{4\cos 2\theta}{\frac{1}{2}(1+\cos^2 2\theta)} = \frac{\cancel{2}\cdot\delta/(\Omega^2+\delta^2)^{1/2}}{\frac{1}{2}\cdot\left(1+\frac{\delta^2}{(\Omega^2+\delta^2)}\right)} \cdot \boxed{\frac{\hbar}{2}\frac{\Omega\nabla\Omega}{(\ )^{1/2}}} =$$

$$\frac{\cancel{\ }\hbar\delta\,\Omega\nabla\Omega}{(\Omega^2+\delta^2)\left(1+\frac{\delta^2}{\Omega^2+\delta^2}\right)} = \frac{\cancel{\ }\hbar\delta\,\Omega\nabla\Omega}{\Omega^2+\delta^2+\delta^2} = \frac{\cancel{\ }\hbar\delta\,\Omega\nabla\Omega}{\Omega^2+2\delta^2} =$$

$$\nabla\Omega^2 = \cancel{2}\,\Omega\nabla\Omega$$

$$\frac{\left(\frac{1}{2}\right)\hbar\delta\,\Omega\nabla\Omega}{\delta^2+\Omega^2/2} = \frac{\frac{1}{4}\hbar\delta\,\nabla\Omega^2}{(\delta^2+\Omega^2/2)}$$ ~~[illegible]~~

$$= \frac{\hbar\delta}{4}\frac{\nabla\Omega^2}{\left(\delta^2+\Omega^2/2\right)}$$

this lacks the term in Γ, as expected for the approximation made, ~~[illegible]~~ where Γ does not appear in the dressed level calculation.