HW 6

1) $\xrightarrow{\hbar\omega} \underset{|g\rangle}{\overset{M}{O}} \rightarrow \bar{v}$ becomes $\underset{|e\rangle}{\overset{M}{O}} \rightarrow \bar{v} + \Delta\bar{v}$

a) $\hbar\bar{k} + M\bar{v} = M(\bar{v} + \Delta\bar{v}) \Longrightarrow \boxed{\Delta\bar{v} = \frac{\hbar\bar{k}}{M}}$

$$\hbar\omega + \frac{1}{2}mv^2 + E_g = E_e + \frac{1}{2}m(\bar{v} + \Delta\bar{v})^2$$

$$\hookrightarrow \boxed{\hbar(\omega - \omega_0) = \frac{\hbar^2 k^2}{2m} + \hbar\bar{v}\cdot\bar{k}}$$

b) The Doppler term, $\bar{v}\cdot\bar{k}$, goes to zero for $\bar{v} = 0$ or $\bar{v} \perp \bar{k}$.

Note that even for $\bar{v}\cdot\bar{k} = 0$, the detuning $\delta = \omega - \omega_0$ cannot be zero; since there is $\hbar k$ momentum associated with the photon, the atom must undergo some finite velocity change as a recoil effect. This is clear in the $\frac{\hbar^2 k^2}{2m} = \frac{1}{2m}(P_{photon})$ term.

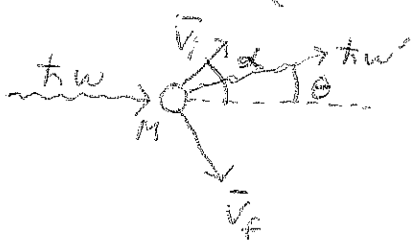
1) c)

Recall only the <u>overall</u> process needs to conserve quantities.

$$M\bar{v}_i + \hbar\bar{k} = M\bar{v}_f + \hbar\bar{k}' \longrightarrow \bar{v}_f = \bar{v}_i + \frac{\hbar}{M}(\bar{k}-\bar{k}')$$

$$\cancel{E_g} + \hbar\omega + \tfrac{1}{2}Mv_i^2 = \cancel{E_g} + \hbar\omega' + \tfrac{1}{2}Mv_f^2$$

so $$\omega' = \omega + \frac{M}{2\hbar}\left(v_i^2 - v_i^2 - \frac{2\hbar}{M}\bar{v}_i\cdot(\bar{k}-\bar{k}') - \frac{\hbar^2}{M^2}(\bar{k}-\bar{k}')^2\right)$$



$$\bar{v}_i\cdot\bar{k} = |v||k|\cos\alpha$$
$$\bar{v}_i\cdot\bar{k}' = |v||k'|\cos(\theta-\alpha)$$
$$\bar{k}\cdot\bar{k}' = |k||k'|\cos\theta$$

note $|k'| = \frac{\omega'}{c}$, $|k| = \frac{\omega}{c}$

$$0 = \omega'^2\left(\frac{\hbar}{2Mc^2}\right) + \omega'\left(1 - \frac{v}{c}\cos(\theta-\alpha) - \frac{\hbar\omega}{Mc^2}\cos\theta\right) + \overbrace{\left(\frac{v\omega}{c}\cos\alpha + \frac{\hbar\omega^2}{2Mc^2} - \omega\right)}^{= -\omega_0 \text{ (part a)}}$$

All quantities are known except θ, ω'. Can find $\omega'(\theta)$ by solving above equation with the quadratic formula. It's a very big expression, though.

$$\boxed{\omega' = \frac{Mc^2}{\hbar}\left[\frac{\hbar\omega}{Mc^2}\cos\theta + \frac{v}{c}\cos(\theta-\alpha) - 1 \pm \sqrt{\left(1-\frac{v}{c}\cos(\theta-\alpha) - \frac{\hbar\omega}{Mc^2}\cos\theta\right)^2 + \frac{2\hbar}{Mc^2}\omega_0}\,\right]}$$

2) a) Since Doppler shift is only relevant for v-components parallel to the light field, set laser in $\hat{z}$ and use 1-D Maxwell distribution:

$$g(v_z) \propto e^{-\frac{m}{2kT}v^2} = e^{-\frac{1}{2}\frac{v^2}{v_{th}^2}}$$

There is a second process broadening the transition:

$$f(v) \propto \frac{\Gamma}{\Gamma^2 + (\delta - \frac{\omega}{c}v)^2}$$ (Lorentzian atomic line w/ Doppler term)

Combining them!

$$\boxed{P(\omega) \propto \int dv \frac{\Gamma}{\Gamma^2 + (\delta - \frac{\omega}{c}v)^2} e^{-\frac{1}{2}\frac{v^2}{v_{th}^2}}}$$

b) note: $\Gamma \propto$ characteristic width of the Lorentzian part.
$v_{th} \propto$ characteristic width of the Gaussian part.

i) $\Gamma << v_{th} k$
Integral is over a Gaussian much broader than the Lorentzian → can treat Lorentzian as a sharp δ-function, giving a Gaussian $P(\omega)$.

ii) $\Gamma >> v_{th} k$
Here the Lorentzian is much broader than the Gaussian → treat Gaussian as a sharp δ-function, giving a Lorentzian $P(\omega)$.

c) Na → $T = 373$ K, $\lambda = 589 \times 10^{-9}$, $\Gamma = 2\pi \cdot 10^{10}$ Hz

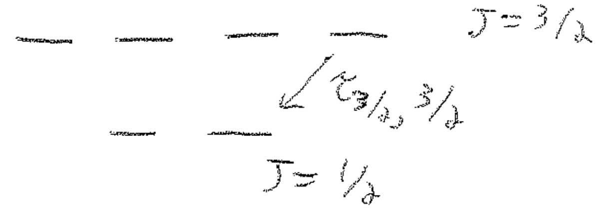
$$v_{th} \simeq \sqrt{\frac{k_B T}{M}} = \sqrt{\frac{1.38 \times 10^{-23} \cdot 373}{23 \cdot 1.66 \times 10^{-27}}} \sim 367 \text{ m/s}$$

$$k = \frac{\omega}{c} = \frac{2\pi}{\lambda} = 1.067 \times 10^{7}$$

$\Gamma = 6.28 \times 10^{11}$ ← factor of 160 different
$k v_{th} = 3.92 \times 10^{9}$

This should result in a fairly Lorentzian lineshape.

3) $J = 3/2,\ J = 1/2$



a) $E_{3/2} > E_{1/2}$

i) From Fermi: $\dfrac{1}{\tau_{3/2\,3/2}} \propto \left|\langle J'M' | d\cdot E | JM \rangle\right|^2$

Note $d\cdot E$ can be written as T^1_q, $q = m' - m$

Wigner-Eckart:

$$\langle J'M' | T^k_q | JM \rangle = \underbrace{\frac{\langle J' \| T^k \| J \rangle}{\sqrt{2J+1}}} \langle Jk;\ Mq \,|\, Jk;\ J'M' \rangle$$

since $J = \frac{3}{2}$, $J' = \frac{1}{2}$ for this problem, this is a constant $\rightarrow C$

$$\tau_{3/2,\,3/2} = \frac{1}{C^2}\cdot\left(\frac{1}{\frac{1}{\sqrt{2}}}\right)^2 = \frac{2}{C^2}$$

$$\frac{1}{\tau_{3/2,\,1/2}} = \frac{1}{\begin{pmatrix}\tau_{3/2,\,1/2} \\ \text{to } \frac{1}{2}, \frac{1}{2}\end{pmatrix}} + \frac{1}{\begin{pmatrix}\tau_{3/2,\,1/2} \\ \text{to } \frac{1}{2}, -\frac{1}{2}\end{pmatrix}}$$

Clebsch-Gordan coefficients

$$= C^2 \cdot \langle \tfrac{3}{2}, 1; \tfrac{1}{2}, 0 \,|\, \tfrac{3}{2}, 1; \tfrac{1}{2}, \tfrac{1}{2} \rangle^2 + C^2 \cdot \langle \tfrac{3}{2}, 1; \tfrac{1}{2}, -1 \,|\, \tfrac{3}{2}, 1; \tfrac{1}{2}, -\tfrac{1}{2} \rangle^2$$

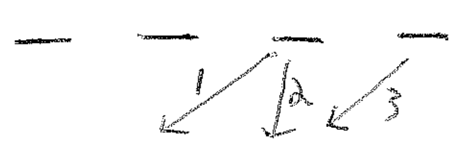
$$= C^2\left(-\frac{1}{\sqrt{3}}\right)^2 + C^2\left(\frac{1}{\sqrt{6}}\right)^2$$

$$= \frac{1}{3}C^2 + \frac{1}{6}C^2$$

$$\tau_{3/2,\,1/2} = \frac{2}{C^2} \implies \boxed{\tau_{3/2,\,1/2} = \tau_{3/2,\,3/2}}$$

ii) From part (i), $\tau_{3/2,\,1/2 \to \frac{1}{2},\frac{1}{2}} = \dfrac{3}{C^2}$, $\tau_{3/2,\,1/2 \to \frac{1}{2},-\frac{1}{2}} = \dfrac{6}{C^2}$

rate $= \dfrac{C^2}{3}$, $\dfrac{C^2}{6}$



$\leftarrow$ ratios of the transition rates

3) b)

i) This is done as before,
Note rates are the same regardless of the direction: $|\langle e|d\cdot E|g\rangle|^2 = |\langle g|d\cdot E|e\rangle|^2$

so $\tau'_{\frac{1}{2},\frac{1}{2}\to\frac{3}{2},\frac{3}{2}} = \tau_{\frac{3}{2},\frac{3}{2}\to\frac{1}{2},\frac{1}{2}} = \tau_{\frac{3}{2},\frac{3}{2}}$ from part (a),

$$\tau'_{\frac{1}{2},\frac{1}{2}\to\frac{3}{2},\frac{1}{2}} = \tau_{\frac{3}{2},\frac{1}{2}\to\frac{1}{2},\frac{1}{2}} = \frac{3}{2}\tau_{\frac{3}{2},\frac{3}{2}}$$

$$\tau'_{\frac{1}{2},\frac{1}{2}\to\frac{3}{2},-\frac{1}{2}} = \tau_{\frac{3}{2},\frac{1}{2}\to\frac{1}{2},-\frac{1}{2}} = 3\,\tau_{\frac{3}{2},\frac{3}{2}}$$

$$\frac{1}{\tau'_{\frac{1}{2},\frac{1}{2}}} = \frac{1}{\tau_{\frac{3}{2},\frac{3}{2}}}\left(1 + \frac{2}{3} + \frac{1}{3}\right) \to \boxed{\tau'_{\frac{1}{2},\frac{1}{2}} = \frac{1}{2}\tau_{\frac{3}{2},\frac{3}{2}}}$$

ii)