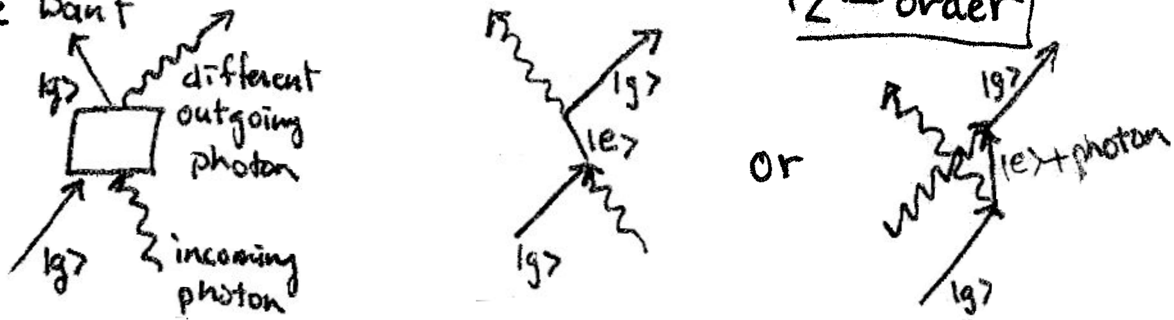
HW #3

1) a) $\vec{d}\cdot\vec{E}$ is odd in atomic space, and involves single photon changing events. We want to have an <u>even</u> atom operator with 2 photons involved: $(\vec{d}\cdot\vec{E})^2$ <u>could do it</u>

we want

$\boxed{2^{nd}\text{ order}}$



1 b) $$V_{gg} = \sum_i \frac{\langle g; n_{k\lambda}-1, 1_{k'\lambda'}|\hat{d}\cdot\hat{E}|i\rangle\langle i|\hat{d}\cdot\hat{E}|g; n_{k\lambda}, 0\rangle}{E_{|g;n_{k\lambda}\rangle} - E_i}$$

Where the energy E_i includes the atom and photon energies of state $|i\rangle$

→ the only photon states coupled by $\hat{E}$ to $|n_{k\lambda}, 0\rangle$ are $|n_{k\lambda}-1, 0\rangle$ and $|n_{k\lambda}, 1_{k'\lambda'}\rangle$

→ the only atom states (in our 2-level problem) coupled by $\hat{d}$ to $|g\rangle$ is $|e\rangle$

$E_0 = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}$

(total energy conservation will require $|k| = |k'|$)

so

$$|i\rangle = |e; n_{k\lambda}, 1_{k'\lambda'}\rangle \quad \text{or} \quad |e; n_{k\lambda}-1, 0\rangle$$

$$V_{gg} = \frac{\langle g|\hat{\vec{d}}|e\rangle\cdot\vec{\epsilon}_{k'\lambda'}\,\langle e|\hat{\vec{d}}|g\rangle\cdot\vec{\epsilon}_{k\lambda}\sqrt{n_{k\lambda}-1}\,E_0^2}{\hbar\omega - \hbar\omega_0} + \frac{\langle g|\hat{\vec{d}}|e\rangle\cdot\vec{\epsilon}_{k\lambda}\,\langle e|\hat{\vec{d}}|g\rangle\cdot\vec{\epsilon}_{k'\lambda'}\sqrt{n_{k\lambda}-1}}{-\hbar\omega_0 - \hbar\omega}$$

$$= \frac{e^2(\vec{r}_{ge}\cdot\vec{\epsilon}_{k'\lambda'})(\vec{r}_{eg}\cdot\vec{\epsilon}_{k\lambda})\sqrt{n_{k\lambda}-1}\,E_0^2}{\hbar\omega - \hbar\omega_0} + \frac{e^2(\vec{r}_{ge}\cdot\vec{\epsilon}_{k\lambda})(\vec{r}_{eg}\cdot\vec{\epsilon}_{k'\lambda'})\sqrt{n_{k\lambda}-1}\,E}{-\hbar\omega_0 - \hbar\omega}$$

1) b) $$V_{gg} = \frac{e^2\omega\sqrt{n_{k\lambda}-1}}{2\epsilon_0 V}\left(\frac{(\vec{r}_{ge}\cdot\vec{\epsilon}_{k'\lambda'})(\vec{r}_{eg}\cdot\vec{\epsilon}_{k\lambda})}{\omega-\omega_0} - \frac{(\vec{r}_{ge}\cdot\vec{\epsilon}_{k\lambda})(\vec{r}_{eg}\cdot\vec{\epsilon}_{k'\lambda'})}{\omega+\omega_0}\right)$$

note: since $\vec{r}_{eg}$ represents an off diagonal matrix element, it doesn't have to be real, just satisfy $\vec{r}_{eg} = \vec{r}_{ge}^{\,*}$. Since $\hat{\vec{r}}$ itself is real, however, then if $\langle e|\hat{\vec{r}}|g\rangle$ is complex, it comes from $|e\rangle$ or $|g\rangle$ being complex, and it is common to choose the wavefunctions to be real, in which case $\vec{r}_{ge} = \vec{r}_{eg}$

$$V_{gg} = \frac{e^2\omega\sqrt{n_{k\lambda}-1}}{2\epsilon_0 V}(\vec{r}_{ge}\cdot\vec{\epsilon}_{k'\lambda'})(\vec{r}_{ge}\cdot\vec{\epsilon}_{k\lambda})\left[\frac{1}{\omega-\omega_0} - \frac{1}{\omega+\omega_0}\right]$$

1) c) $$R = \int\sum_{\lambda'} d\Omega_{k'}\left(\frac{2\pi}{\hbar}\right)\frac{e^4\omega^2(n_{k\lambda}-1)}{4\epsilon_0^2 V^2}|\vec{r}_{ge}\cdot\vec{\epsilon}_{k'\lambda'}|^2|\vec{r}_{g e}\cdot\vec{\epsilon}_{k\lambda}|^2\left|\frac{1}{\omega-\omega_0} - \frac{1}{\omega+\omega_0}\right|^2\rho(\hbar\omega)$$

$$\sum_{\lambda'}\int d\Omega_{k'}\,|\vec{r}_{ge}\cdot\vec{\epsilon}_{k'\lambda'}|^2 = 2\times\frac{4\pi}{3}|r_{ge}|^2 \quad , \quad \rho(\hbar\omega) = \frac{V}{(2\pi)^3}\frac{\omega^2}{\hbar c^3}$$

$$= \left(\frac{2\pi}{\hbar^2}\right)\frac{e^4\omega^4(n_{k\lambda}-1)}{4\epsilon_0^2 c^3 V(2\pi)^3}\left(\frac{8\pi}{3}\right)\underbrace{C_{eg}^2|r_{eg}|^4}_{|\vec{r}_{eg}\cdot\vec{\epsilon}_{k\lambda}|^2 \equiv C_{eg}^2|\vec{r}_{eg}|^2}\left(\frac{1}{\omega_0-\omega} + \frac{1}{\omega+\omega_0}\right)^2$$ ← pull out a minus sign

$$= e^4|r_{eg}|^4 C_{eg}^2\underbrace{\left(\frac{\hbar\omega n}{V}c\right)}_{I}\frac{\omega^3}{6\pi\epsilon_0^2\hbar^3 c^4}\left|\frac{1}{\omega_0-\omega} + \frac{1}{\omega+\omega_0}\right|^2$$

$(n \gg 1, (n-1)\simeq n)$

$$\Gamma = \frac{e^2|r_{eg}|^2\omega_0^3}{3\pi\hbar\epsilon_0 c^3}$$

$$= I\,\Gamma^2\, C_{eg}^2\left(\frac{\omega}{\omega_0}\right)^3\frac{3\pi}{2}\frac{c^2}{\hbar}\frac{1}{\omega_0^3}\left|\frac{1}{\omega_0-\omega} + \frac{1}{\omega+\omega_0}\right|^2$$

1) c) $\frac{\hbar \omega_0^3}{c^2}$ has units of an Intensity per unit angular frequency. Turn this into an intensity by multiplying / dividing by Γ

define $I_{sat} = \left(\frac{\Gamma}{4}\right)\frac{\hbar \omega_0^3 C_{eg}^2}{3\pi c^2}$ $\longrightarrow$ other options possible, but this is the standard def.

$$R = \frac{1}{8}\left(\frac{I}{I_{sat}}\right)\Gamma^3 \left|\frac{1}{\omega_0-\omega} + \frac{1}{\omega+\omega_0}\right|^2 \left(\frac{\omega}{\omega_0}\right)^3$$

i) $\omega << \omega_0$

$$R = \frac{1}{2}\left(\frac{I}{I_{sat}}\right)\frac{\Gamma^3}{\omega_0^2}\left(\frac{\omega}{\omega_0}\right)^3 \quad \propto \omega^3 \text{ Rayleigh scattering}$$

ii) $\omega >> \omega_0$ $\quad \frac{1}{\omega_0 \pm \omega} \simeq \frac{1}{\omega}\left(\pm 1 - \frac{\omega_0}{\omega} + \dots\right)$

$$R = \frac{1}{8}\frac{I}{I_0}\Gamma^3\left(\frac{2\omega_0}{\omega^2}\right)^2\left(\frac{\omega}{\omega_0}\right)^3 = \frac{1}{2}\frac{I}{I_0}\frac{\Gamma^3}{\omega\,\omega_0}$$

iii) $\omega \simeq \omega_0$

$$R = \frac{1}{8}\frac{I}{I_0}\Gamma^3\left|\frac{1}{\omega_0-\omega}\right|^2 \underbrace{\left(\frac{\omega}{\omega_0}\right)^3}_{\simeq 1}$$

$$= \frac{1}{8}\frac{I}{I_0}\frac{\Gamma^3}{|\omega_0-\omega|^2} \Rightarrow \text{diverges as } \omega \to \omega_0$$

1)d) $E_e \to E_e - i\hbar\Gamma/2 \Rightarrow$ replace ω_0 by $\omega_0 - \frac{i\Gamma}{2}$

$\omega \simeq \omega_0$

$$R = \frac{1}{8}\frac{I}{I_0}\underbrace{\left(\frac{\omega}{\omega_0}\right)^3}_{\simeq 1}\frac{\Gamma^3}{\left|\omega_0-\omega-\frac{i\Gamma}{2}\right|^2}$$

$\Delta = \omega_0 - \omega$

$$R = \frac{1}{8}\frac{I}{I_0}\frac{\Gamma^3}{\Delta^2 + \Gamma^2/4}$$

$R(\Delta=0) = \frac{1}{2}\Gamma\left(\frac{I}{I_0}\right) \to$ this is just $\frac{\Gamma}{2}$ times the normalized incident intensity.

1 e) the real part of E gives rise to a quantum mechanically well defined energy $\Rightarrow$ doesn't have decay built in. The excited state energy is <u>not</u> well defined, and the excited state <u>amplitude</u> in fact decays at $\Gamma/2$. Phenomenologically replacing E_e by $E_e - i\hbar\frac{\Gamma}{2}$ mimics the effect of decay on spreading the energy width. (although it is technically not Unitary time evolution)