Homework #2 Soln.

1) a) Note: this problem was intentionally vague, to give you a chance to play around with orders of magnitude. It will not be weighted very heavily for the grade. (Since there are many ways to describe the answer)

$$H_{ed} \simeq e a_0 E$$

$$H_A \simeq \frac{-1}{4\pi\epsilon_0}\frac{e^2}{2a_0} \text{ (Hydrogen) or } \simeq -\frac{1}{4\pi\epsilon_0}\frac{Z^2 e^2}{2a_0} \text{ for larger atoms.}$$

Focus on scalings for Hydrogen-

$$\left|\frac{H_{ed}}{H_A}\right| \simeq \frac{e a_0 E}{\frac{1}{4\pi\epsilon_0}\frac{e^2}{2a_0}} = \frac{e a_0}{e a_0}\left(\frac{\overbrace{E}^{\text{applied field}}}{\underbrace{\frac{e}{4\pi\epsilon_0 2a_0^2}}_{\text{field inside the atom}}}\right)$$

$$E_A \equiv \frac{e}{4\pi\epsilon_0 2a_0^2} \simeq 3\times10^{11}\frac{V}{m}, \quad 300\frac{MV}{m}$$

→ this is a Huge field. Any applied E is likely to be small compared to this.

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b)

$$H_{\mu_L} \simeq \mu_L B \cong \frac{e}{2m} l \frac{E}{c} \qquad l \cong \hbar$$

$$H_{\mu_S} \simeq \mu_S B \cong \frac{e}{m} s \frac{E}{c} \qquad s \simeq \hbar/2$$

$$H_{\mu, s} \simeq \frac{e\hbar}{2m}\frac{E}{c}$$

1.) b) cont.

$$\frac{H_{\mu_{L,S}}}{H_{ed}} \simeq \frac{\frac{e\hbar}{2mc} E}{e a_0 E} = \frac{\hbar}{2mc a_0} \qquad a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m e^2} \simeq 0.5\times10^{-10}\,m$$

$$= \frac{e^2}{4\pi\epsilon_0 \hbar c} \equiv \boxed{\alpha \simeq \frac{1}{137}}$$

$\frac{H_{\mu_{L,S}}}{H_{ed}}$ is independent of E, and depends only on fundamental constants, expressed in the dimensionless constant α, the fine structure constant. (doesn't even depend on the mass.)

$$H_Q \simeq e a_0^2 \nabla E = e a_0^2 k E,$$ k is the wave vector of light, $k = \frac{\omega}{c}$

$$\frac{H_Q}{H_{ed}} \cong \frac{e a_0^2 \omega/c\, E}{e a_0 E} = \frac{a_0 \omega}{c}$$

For approximately resonant light, $\hbar\omega \simeq \frac{e^2}{4\pi\epsilon_0 2a_0}$

$$\boxed{\cong \frac{e^2}{4\pi\epsilon_0 \hbar c} = \alpha \simeq 1/137}$$

its the same order of magnitude as the magnetic dipole, also independent of E. Also note: transition rates go as $|\langle H_I \rangle|^2$

so rates depend on $\alpha^2 \to$ pretty small.

1)b)

$$H_{B^2} \simeq \frac{e^2}{2m} a_0^2 \frac{E^2}{c^2} \qquad B^2 = \frac{E^2}{c^2}$$

$$\frac{H_{B^2}}{H_{ed}} \simeq \frac{\frac{e^2}{2m} a_0^2 \, E^2/c^2}{e a_0 E} = \frac{e}{2mc^2} a_0 E$$

$$\simeq \frac{e a_0 E_A}{2mc^2}\left(\frac{E}{E_A}\right)$$

E_A is the electric field in the atom, defined in 1.a)

$\left(\frac{E}{E_A}\right)$ → depends on E!

$\left(\frac{E}{E_A}\right)$ → $= \left(\frac{H_{ed}}{H_A}\right)$

$e a_0 E_A$ → electrostatic potential in the atom

mc^2 → rest energy of electron

$$e a_0 E_A \simeq 13 \text{ eV}$$

$$mc^2 \simeq 500 \text{ keV}$$

$$\boxed{\frac{e a_0 E_A}{mc^2} \simeq 3\times10^{-5}}$$

this is <u>very</u> small for two reasons: 1) $\frac{e a_0 E_A}{mc^2}$ is small. 2) $\frac{E}{E_A}$ is usually very small.

another way to write this is: (not showing the algebra)

$$\frac{H_{B^2}}{H_{ed}} = \alpha\left(\frac{V_A}{c}\right)\left(\frac{E}{E_A}\right) \qquad \text{using } V_A = \frac{p}{m} \simeq \frac{\hbar}{m a_0} \qquad \Delta p \simeq \frac{\hbar}{\Delta x}$$

2 a) $\hat{\vec{d}}$ is odd, connects opposite parity states $d \propto \vec{r} \Rightarrow d \propto -\vec{r}$ under inversion

$\hat{\vec{\mu}}_L$ is even, connects same parity states

$$\mu \propto \vec{r} \times \vec{p} \overset{\text{inversion}}{\Rightarrow} \mu \propto -\vec{r} \times (-\vec{p}) = \vec{r} \times \vec{p}.$$

$\overleftrightarrow{Q}$ is even, connects same parity states

$$\overleftrightarrow{Q} \propto \vec{r}\,\vec{r} \overset{\text{inversion}}{\Longrightarrow} Q \propto (-\vec{r})(-\vec{r}) = \vec{r}\,\vec{r}$$

$(\vec{r} \times \vec{B})^2$ is even, connects same parity states

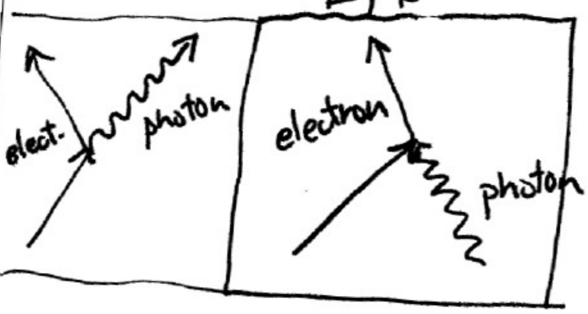
b) $\vec{E} \propto \hat{a} + \hat{a}^{+} \rightarrow$ couples photon states that differ by 1 $\rightarrow$ single photon operator

$\vec{B} \propto \hat{a} - \hat{a}^{+} \rightarrow$ couples photon states that differ by 1 $\rightarrow$ single photon operator

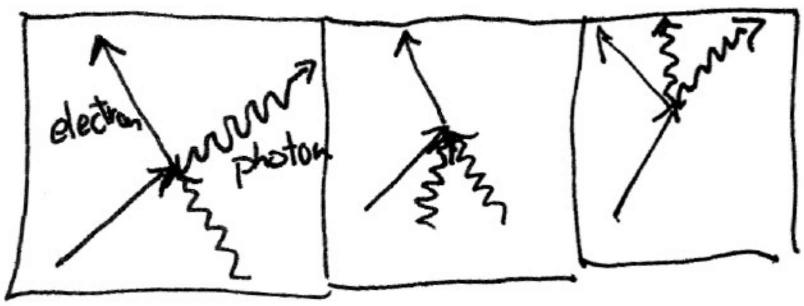
$\vec{B}^2 \propto \hat{a}\hat{a} + \hat{a}\hat{a}^{+} + \hat{a}^{+}\hat{a} + \hat{a}^{+}\hat{a}^{+} \rightarrow$ couples states that differ by 0 photons or 2 photons.

Schematically:

$\hat{\vec{E}}, \hat{\vec{B}}$



$\vec{B}^2$



3) a) For an atom in free space, physics should be rotationally invariant $\Rightarrow$ lifetime cannot depend on the orientation of the atom. (Not true if an external field is applied, because of the defined axis in space, though.) Rotation of the atom mixes the m levels $\rightarrow$ since lifetime is invariant to rotations, it must be the same for all levels.

b)

$$\frac{dR}{\partial \Omega_k} = \frac{2\pi}{\hbar} \left| \langle 1,0,0; 1_{k\lambda} | \vec{d} \cdot \hat{E}_{k\lambda} | 2\,l\,m; 0 \rangle \right|^2 \rho(\hbar\omega_0)$$

similar to derivation of Einstein A coef. in lecture 05.

$$= \frac{2\pi}{\hbar} e^2 \left| \langle 1,0,0 | \vec{r} | 2\,l\,m \rangle \cdot \vec{\mathcal{E}}_{k\lambda} \right|^2 |E_0|^2 \frac{V}{(2\pi)^3} \frac{\omega_0^2}{\hbar c^3}$$

$$|E_0|^2 = \frac{\hbar\omega_0}{2\epsilon_0 V} \quad \text{for } \omega = \omega_0$$

k, λ are the outgoing photon mode.

$$= \frac{e^2 \omega_0^3}{2\epsilon_0 4\pi^2 \hbar c^3} \left| \langle 1,0,0 | \vec{r} | 2\,l\,m \rangle \cdot \vec{\mathcal{E}}_{k\lambda} \right|^2$$

define $\vec{M} = \langle 100 | \vec{r} | 2\,l\,m \rangle$

$$R = \sum_\lambda \int \frac{e^2 \omega_0^3}{2\epsilon_0 4\pi^2 \hbar c^3} |\vec{M} \cdot \vec{\mathcal{E}}_{k\lambda}|^2 \, d\Omega_k$$

3 b). To evaluate the integral of $|\vec{M}\cdot\vec{\epsilon}_{k\lambda}|^2$, choose a coordinate system where $\vec{M}$ is along $\hat{z}$, for example

$$\sum_\lambda \int |\vec{M}\cdot\vec{\epsilon}_{k\lambda}|^2 d\Omega_k = \iint |\vec{M}|^2 \cos^2\theta \sin\theta\, d\theta\, d\phi$$

$\vec{M}\cdot\vec{\epsilon}_{k\lambda} = |\vec{M}|\cos\theta$

$$= 2|\vec{M}|^2 \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \qquad \sum_\lambda 1 = 2$$

$$= 2|\vec{M}|^2\, 2\pi \int_{-1}^{1} u^2 du = 2|\vec{M}|^2\, 2\pi\, \frac{2}{3} = |\vec{M}|^2 \frac{8\pi}{3}$$

So $\boxed{R = \frac{e^2 \omega_0^3}{3\epsilon_0 \pi \hbar c^3} |\vec{M}|^2}$ → exactly what we had in class for A_{eg}.

Now to evaluate $\vec{M}$:

$$\phi_{100} = \frac{e^{-r/a_0}}{\sqrt{\pi}\, a_0^{3/2}} = \frac{2e^{-r/a_0}}{a_0^{3/2}} Y_{0,0}(\theta,\phi)$$

$$\phi_{210} = \frac{1}{4\sqrt{2\pi}}\left(\frac{r}{a_0}\right) e^{-r/2a_0} / a_0^{3/2} \cos\theta = \frac{2}{\sqrt{3}}\left(\frac{r}{2a_0}\right)\frac{e^{-r/2a_0}}{(2a_0)^{3/2}} Y_{1,0}(\theta,\phi)$$

$$\phi_{21\pm1} = \frac{1}{8\sqrt{\pi}}\left(\frac{r}{a_0}\right) \frac{e^{-r/2a_0}}{a_0^{3/2}} \sin\theta\, e^{\pm i\phi} = \frac{2}{\sqrt{3}}\left(\frac{r}{2a_0}\right)\frac{e^{-r/2a_0}}{(2a_0)^{3/2}} Y_{1\pm1}(\theta,\phi)$$

$$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$= r\sqrt{\frac{8\pi}{3}}\left(\frac{-Y_{11}+Y_{1-1}}{2}\right)\hat{x} + r\sqrt{\frac{8\pi}{3}}\left(\frac{-Y_{11}-Y_{1-1}}{2i}\right)\hat{y} + \sqrt{\frac{4\pi}{3}}\, Y_{10}\, r\,\hat{z}$$

3 b)

$$\vec{M} = \int \phi_{100}(\vec{r})\, \vec{r}\, \phi_{21m}(\vec{r})\, d^3r$$

$\vec{r} \propto Y_{lm}$, $l=1$, and $\phi_{21m}(\vec{r}) \propto Y_{lm}$ $l=1$,

choose $\phi_{210}(\vec{r})$ so only the $\hat{z}$ component of $\vec{r}$ will give a non-zero integral:

$$\vec{M} = \hat{z} \iiint \left(\frac{2e^{-r/a_0}}{a_0^{3/2}} \frac{1}{\sqrt{4\pi}} \right) \left(\sqrt{\frac{4\pi}{3}}\, r\, Y_{10}(\theta,\phi) \right) \left(\frac{2}{\sqrt{3}} \left(\frac{r}{2a_0}\right) \frac{e^{-r/2a_0}}{(2a_0)^{3/2}} Y_{10}(\theta,\phi) \right) \times r^2 dr\, d\Omega .$$

$$= \hat{z} \int_0^\infty \frac{2^4}{3^5\sqrt{2}} e^{-\frac{3r}{2a_0}} \left(\frac{3r}{2a_0}\right)^4 dr \qquad \int d\Omega\, Y_{10} Y_{10} = 1$$

$$= \frac{2^5 a_0 \hat{z}}{3^6 \sqrt{2}} \underbrace{\int_0^\infty e^{-u} u^2 du}_{24} = \frac{2^7\sqrt{2}}{3^5} a_0 \hat{z} \simeq \frac{3}{4} a_0 \hat{z}$$

So $$\boxed{R = \frac{e^2 \omega_0^3}{3\epsilon_0 \pi \hbar c^3} \left| \frac{2^7\sqrt{2}}{3^5} \right|^2 a_0^2}$$

This can be written

$$= \frac{3}{16} \omega_0 \alpha^3 \left(\frac{2^{15}}{3^{10}} \right)$$

using: $\hbar\omega_0 = \frac{e^2}{4\pi\epsilon_0} \frac{1}{2a_0}\left(1 - \frac{1}{2^2}\right)$

$$\omega_0 a_0 = \frac{3}{8} \alpha c$$

numerically, $R = \frac{3}{16}(1.55 \times 10^{16} s^{-1}) \frac{1}{137^3} \frac{2^{15}}{3^{10}}$

$R = 6.27 \times 10^8 s^{-1}$

$\boxed{\tau = \frac{1}{R} = 1.59 ns \longrightarrow \text{experimental value } 1.600 ns}$