Discussion of photons brings us to quantized light:

How do we move to quantized fields?

replace
$$\vec{E} \rightarrow \hat{\vec{E}} = \sum_{k\lambda} i\, \mathcal{E}_{k\lambda}\, \hat{\epsilon}_{k\lambda} \left(\hat{a}_k e^{i k \cdot r} - \hat{a}_k^{\dagger} e^{-i k \cdot r} \right)$$

note: some authors subsume i into $\hat{a}$, $(i\hat{a})^{\dagger} = -i\hat{a}^{\dagger}$ $\quad -i^2 \hat{a}^{\dagger}\hat{a} = \hat{a}^{\dagger}\hat{a}$

can also be written
$$\hat{\vec{E}} = \sum_{k\lambda} \mathcal{E}_k\, \hat{\epsilon}_{k\lambda} \left(\hat{a}_k e^{i\vec{k}\cdot\vec{r}} + \hat{a}_k^{\dagger} e^{-i\vec{k}\cdot\vec{r}} \right)$$

$$\vec{E}^{(+)}(\vec{r}) = \sum_{k\lambda} \mathcal{E}_k \hat{\epsilon}_{k\lambda} \hat{a}_k e^{ik r}$$
$$\vec{E}^{-}(\vec{r}) = \sum \hat{a}^{\dagger} e^{-ik\cdot r}$$

Problem: what is the intensity?

$$\frac{1}{2}\epsilon_0 c |E|^2 = c\, \underset{\uparrow \text{energy density}}{u} = c \sum_k \underbrace{\hbar\omega_k \left(\hat{a}_k^{\dagger}\hat{a}_k + 1/2 \right)}_{\text{diverges.}}$$

Look at detection from a quantum point of view:

use photo-excitation/ionization as a model.

starting in state $|i\rangle = |\{n_k\}, \varphi_0\rangle$, where $|\varphi_0\rangle$ is the state of the detector, we detect light when photons make transitions to $|\varphi_f\rangle$

e.g.
$$\langle \varphi_f | \hat{\vec{d}} \cdot \hat{\vec{E}} | \varphi_0 \rangle \rightarrow \text{matrix element.}$$

in any event, the radiative part enters independently of the atomic (detector part) and its contribution to the rate is (for a single mode)

$$\left| \langle n_k - 1 | \hat{E} | n_k \rangle \right|^2$$

Only the destruction operator part contributes,

$$|\langle n_k - 1|\, \hat{E}^{+}(r)\, | n_k \rangle|^2 = |\langle n_k | \hat{E}^{-}(r) | n_k - 1\rangle \langle n_k - 1 | \hat{E}^{+} | n_k \rangle$$

$\hat{E}$ is not a vector, its the magnitude.

$$= \langle n_k | \hat{E}^{(-)}(r)\, \hat{E}^{(+)}(r) | n_k \rangle.$$

$\rightarrow$ called normally ordered.

For a statistical average, ~~this part is~~ The photon contribution to the rate is

$$\sum_{n_{k\lambda}} P(n_{k\lambda}) \langle n_{k\lambda} | \hat{E}^{(-)}(r)\, \hat{E}^{(+)}(r) | n_{k\lambda} \rangle = \mathrm{Tr}\left(\hat{\rho}\, \hat{E}^{-}(r)\, \hat{E}^{+}(r)\right)$$

(the atomic part drops out as well, if you sum over all possible final detector states leaving you with an overall quantum efficiency)

this can be generalized to a quantum-optical pointing vector

$$\hat{\vec{I}} = \epsilon_0 c^2 \left(\hat{\vec{E}}^{(-)}(r) \times \hat{\vec{B}}^{(+)}(r) - \hat{\vec{B}}^{(-)}(r) \times \hat{\vec{E}}^{(+)}(r) \right)$$

which for polarized parallel light gives

$$\hat{I} = 2\epsilon_0 c\, \hat{E}^{(-)}(r)\, \hat{E}^{(+)}(r)\, \frac{\vec{k}}{|\vec{k}|}$$

and

$$\langle \hat{I} \rangle = \frac{c}{V} \sum_k \hbar\omega_k \langle n_{k\lambda} \rangle = \frac{c}{V}\left(\langle E_R \rangle - E_{vac} \right)$$

Use Heisenberg operators

$$g^{(1)}(\vec{r}_1,\vec{r}_2,\tau) = \frac{\langle \hat{E}^{(-)}(r_1) \hat{E}^{(+)}(r_2) \rangle}{\sqrt{\langle \hat{E}^{(-)}(r_1)\hat{E}^{(+)}(r_1)\rangle \langle \hat{E}^{(-)}(r_2)\hat{E}^{(+)}(r_2)\rangle}}$$

$$g^{(2)}(r_1,r_2,\tau) = \frac{\langle \hat{E}^{(-)(t)}(r_1)\, \hat{E}^{-(t+\tau)}(r_2)\, \hat{E}^{(+)(t+\tau)}(r_2)\, \hat{E}^{(+)(t)}(r_1) \rangle}{\langle \hat{E}^{-}(r_1)\hat{E}^{+}(r_1)\rangle \langle \hat{E}^{-}(r_2)\hat{E}^{+}(r_2)\rangle}$$

where $\langle \hat{O} \rangle = \mathrm{Tr}(\hat{\rho}\hat{O})$

$$\rightarrow \langle I_{MZ} \rangle = \frac{1}{2}\epsilon_0 \frac{1}{4}\left(\langle (\hat{E}^{-}(t_1) + \hat{E}^{-}(t_2))(\hat{E}^{(+)}(t_1) + \hat{E}^{(+)}(t_2)) \rangle \right.$$

One can show $0 \leq g^{(2)}(\tau) \leq \infty$, but one <u>cannot</u> show that $g^{(2)}(0) \geq 1$!

Examples → start with single mode only, for examples

→ note: single mode is <u>monochromatic</u>.

classically we saw that

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, g^{(1)}(\tau)\, e^{i\omega\tau}$$

$\rightarrow g^{(1)}(\tau) \rightarrow$ constant.

→ light is a sine-wave of fixed phase, ϕ

pretty boring deterministic light.

ϕ is the only variable freedom

Scale electric field operator so that

$$\hat{E}(\phi) = \vec{E}^+(\phi) + \vec{E}^-(\phi) = \frac{1}{2}\hat{a}e^{-i\phi} + \frac{1}{2}\hat{a}^+ e^{i\phi}$$

$$\rightarrow = \hat{X}\cos\phi + \hat{Y}\sin\phi \qquad \phi = (\omega t - \vec{k}\cdot\vec{r})$$

quadrature expression

$$\hat{X} = \frac{\hat{a}+\hat{a}^+}{2} \quad \hat{Y} = \frac{\hat{a}-\hat{a}^+}{2i}$$

- $\left[\hat{E}(\phi_1), \hat{E}(\phi_2)\right] = \frac{-i}{2}\sin(\phi_1-\phi_2)$

↳ $\hat{E}$ at different ϕ don't commute.

For single-mode

$$g^{(1)}(x_1,x_2,\tau) = \frac{\langle \frac{1}{2}\hat{a}^+ e^{i\phi_2} \frac{1}{2}\hat{a}e^{-i\phi_1}\rangle}{\sqrt{\langle \frac{1}{4}\hat{a}^+\hat{a}\rangle\langle \frac{1}{4}\hat{a}^+\hat{a}\rangle}}$$

$$= e^{i(\phi_2-\phi_1)} \qquad \phi_1 = -kx_1 + \omega t$$

$$\phi_2 = -kx_2 + \omega(t+\tau)$$

$$|g^{(1)}(\tau)| = \underline{1} \rightarrow \text{kind of expected.}$$

define $S(\hat{E}(\phi)) = \langle \hat{E}(\phi)\rangle$

$$N(E(\phi)) = (\Delta E(\phi))^2 \qquad (\Delta E(\phi))^2 = \langle \hat{E}^2\rangle - \langle E\rangle^2$$

What about $g^{(2)}(\tau)$?

$$g^{(2)}(\cancel{x_1} x_2, \tau) = \frac{\frac{1}{8}\langle \hat{a}^\dagger e^{i\phi_1} \hat{a}^\dagger e^{i\phi_2} \hat{a} e^{-i\phi_2} \hat{a} e^{-i\phi_1} \rangle}{\langle \frac{\hat{a}^\dagger \hat{a}}{4} \rangle \langle \frac{\hat{a}^\dagger \hat{a}}{4} \rangle}$$

$$= \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2}$$

Note: $g^{(2)}$ doesn't depend on $\Delta\phi$, (or Δx or τ),

but it does depend on the state that is chosen, and it is not always given by deterministic light!

$$\hat{a}^\dagger \hat{a} = \hat{a}\hat{a}^\dagger - 1$$

$$g^{(2)}(\tau) = \frac{\langle n(n-1) \rangle}{\langle n \rangle^2} = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2}$$

add to subtract 1

$$= 1 + \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle^2}$$

Since $(\Delta n)^2 \geq 0$ $\qquad (\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2$

$$g^{(2)}(\tau) \geq 1 - \frac{1}{\langle n \rangle}$$

$g^{(2)}(\tau) \geq 0$ no matter what.

for finite photon number, this is less than 1! Statistically impossible for classical light.

Compare to EPR!

Specific examples:

number or Fock state:

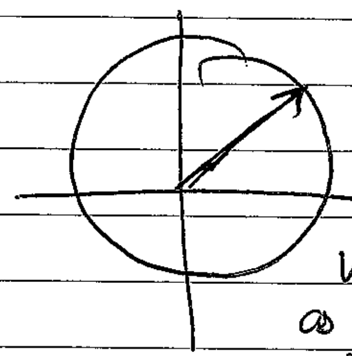
$$(\Delta n)^2 = 0, \qquad g^{(2)} = 1 - \frac{1}{n} \quad \text{for } n \geq 1$$

we showed early in the class, that the quadrature expectation values on number states vanish:

$$\langle n | \hat{X} | n \rangle = \langle n | \hat{Y} | n \rangle = 0$$

but

$$(\Delta X)^2 = (\Delta Y)^2 = \tfrac{1}{2}(n + 1/2)$$



viewed as all phases available.

Coherent states:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*$$

$$|\langle\alpha|\beta\rangle|^2 \neq 0 = e^{-|\alpha-\beta|^2} \rightarrow \text{overcomplete.}$$

$$\langle n \rangle = \langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2$$

$$\langle n^2 \rangle = \langle\alpha|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}|\alpha\rangle$$

$$= |\alpha|^4 + |\alpha|^2 = \langle n \rangle^2 + \langle n \rangle$$

$$(\Delta n)^2 = \langle n \rangle$$ $\rightarrow$ not like a deterministic classical wave.

$\frac{\Delta n}{\langle n\rangle} \approx \frac{1}{\sqrt{\langle n\rangle}}$ relative uncertainty,

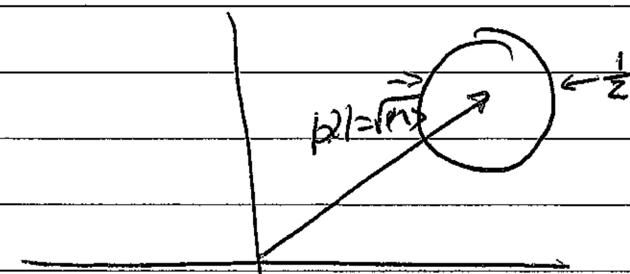
not surprising $P(n)$ is poissonian

$$|\langle n|\alpha\rangle|^2 = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!}$$

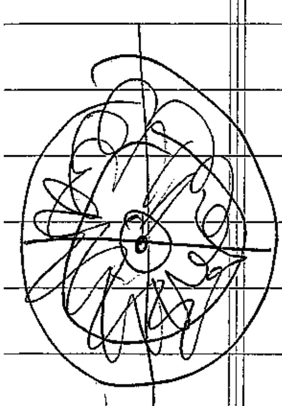
Nonetheless,

$$g^{(2)}(\tau) = \frac{\langle\alpha|\hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a}|\alpha\rangle}{(\langle\alpha|\hat{a}^\dagger \hat{a}|\alpha\rangle)^2} = 1 + \underbrace{\frac{(\Delta n)^2 - \langle n\rangle}{\langle n\rangle^2}}_{=0}$$

$$g^{(2)}(\tau) = 1 \rightarrow \text{no "bunching"}$$



Chaotic light (single mode)



Thermal distribution of photon number (earlier in class)

$$P(n) = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{1+n}}$$

$$(\Delta n)^2 = \langle n\rangle^2 + \langle n\rangle$$

$$g^{(2)}(\tau) = 2 \text{ for } \langle n\rangle \text{ large.}$$

definitions:

if $(\Delta n)^2 - \langle n\rangle > 0 \Rightarrow$ super poissonian

$(\Delta n)^2 - \langle n\rangle < 0$ sub poissonian

Quadrature.

Squeezed light:

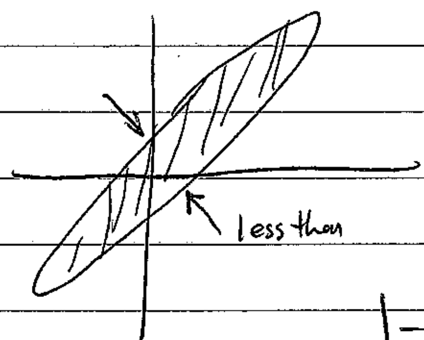
Consider the state $|s\rangle = \hat{S}(s)|0\rangle$ → squeezed vacuum

$$\hat{S}(s) = e^{\left(\frac{1}{2}s^*\hat{a}^2 - \frac{1}{2}s(\hat{a}^\dagger)^2\right)}, \quad s = |s|e^{i\theta}$$

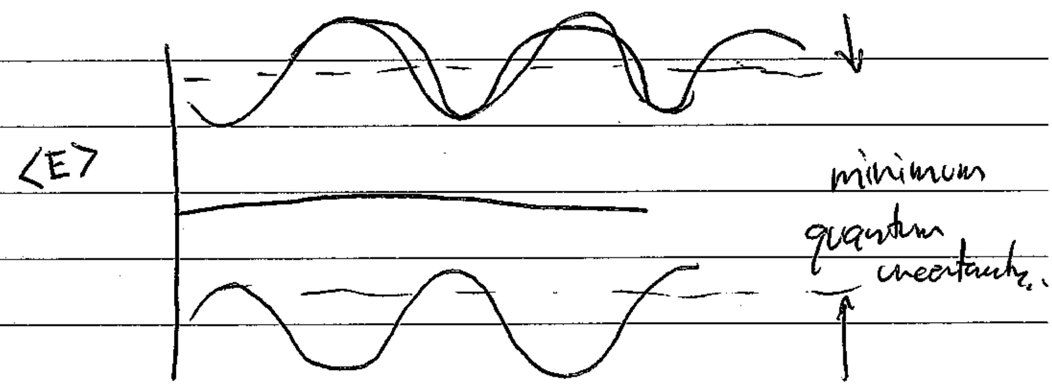
→ looks like a coherent state, but $\hat{a}^2, (\hat{a}^\dagger)^2$
(immediately indicates needed non-linear processes in materials, not E, but $\hat{E}^2$

can be shown that $g^{(2)}(\tau) = 3 + \frac{1}{\langle n\rangle}$

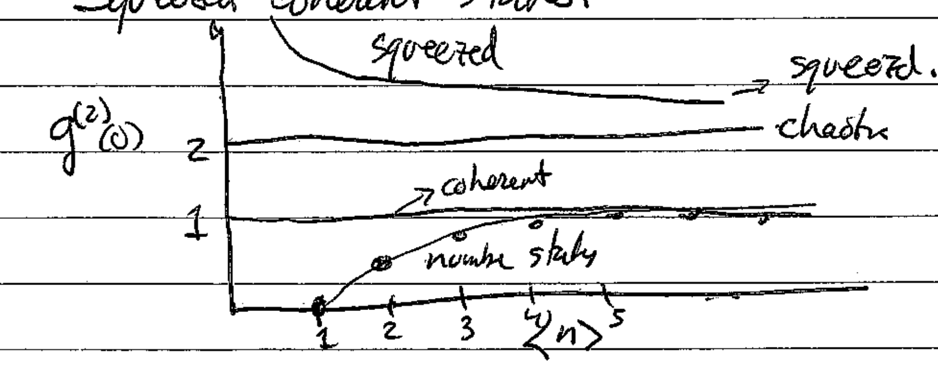
↳ super poisson



$\langle n\rangle \neq 0 = \sinh^2|s|$



Squeezed coherent states.



First real indication for the need for photons!
(not photoelectric effect)

To deal with multiple modes (e.g. different ports on a beam splitter) or multiple frequencies (pulses of light e.g.) one needs multimode or continuous mode formalisms.

Multimode: add a $\vec{k}$ and $\omega_{\vec{k}}$ dependence,

$$[\hat{a}_{\vec{k}}, \hat{a}^{\dagger}_{\vec{k}'}] = \delta_{kk'}, \qquad |\Psi\rangle = |\psi_{k_1}\rangle|\psi_{k_2}\rangle|\psi_{k_3}\rangle \cdots |\psi_{k_n}\rangle$$

→ get very similar results for "classical"-like light.

Continuous mode: frequency spectrum continuous: imagine a box in space (size L) with frequency spacing $\Delta\omega = 2\pi c/L$ in the limit that $L \to \infty$, $\Delta\omega \to 0$

example for co-propagating beems.

$$\sum_k \to \frac{1}{\Delta\omega}\int d\omega \qquad \delta_{kk'} \to \Delta\omega\, \delta(\omega-\omega')$$

$$\hat{a}_k \to \sqrt{(\Delta\omega)}\, \hat{a}(\omega)$$

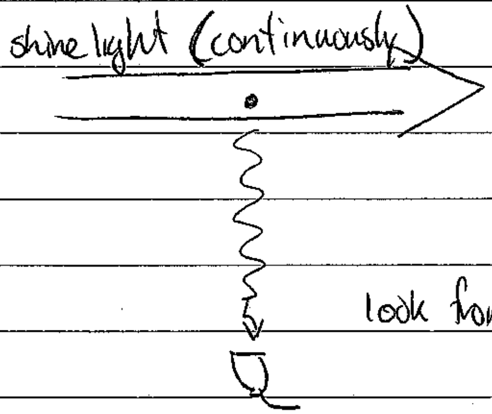
$$[\hat{a}(\omega), \hat{a}^{\dagger}(\omega')] = \delta(\omega-\omega')$$

e.g.

$$H_R = \sum_k \hbar\omega_k (\hat{a}^{\dagger}_k \hat{a}_k) + \text{Vacuum} \longrightarrow \int_0^{\infty} d\omega\, \hbar\omega\, \hat{a}^{\dagger}(\omega)\, \hat{a}^{\dagger}(\omega) + \text{vacuum}.$$

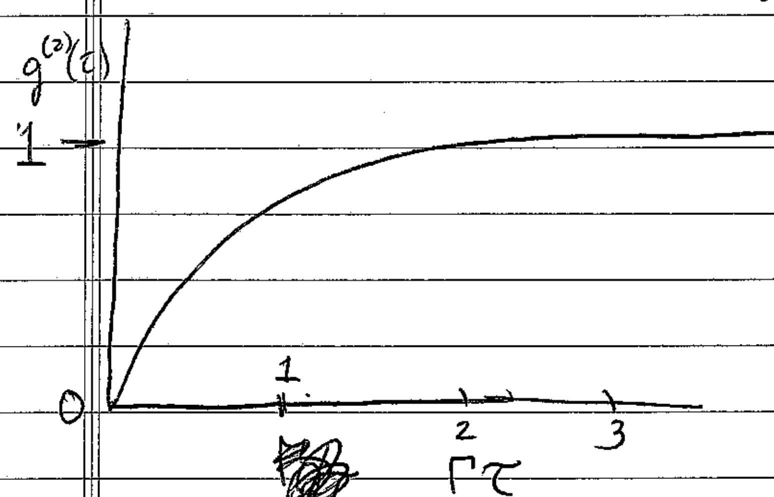
$\hat{n} = \int d\omega\, \hat{a}^{\dagger}(\omega)\hat{a}(\omega)$, can define $\hat{a}(t) = \int_{-\infty} d\omega \frac{\hat{a}(\omega) e^{-i\omega t}}{\sqrt{2\pi}}$

Resonance Flouresence from a single atom.



classically, the radiation is deterministic sine-wave.
quantum mechanically: anti bunching, sub poissonian light.

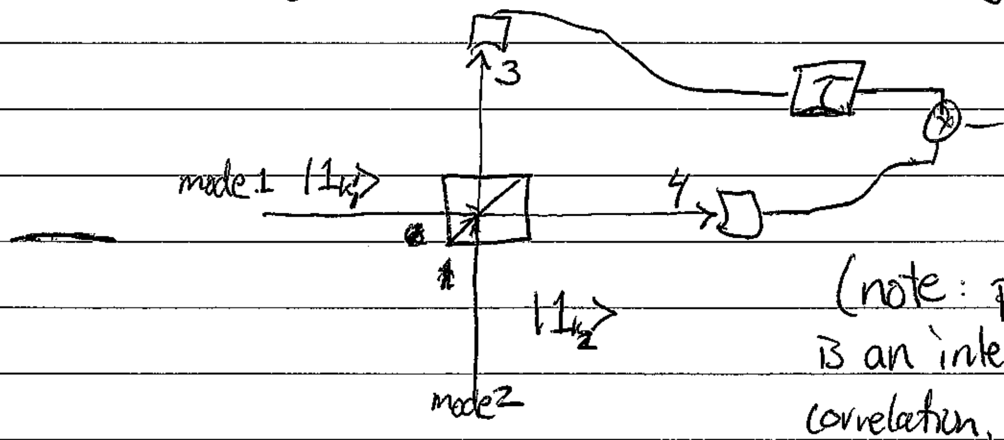
$$g^{(2)}(\tau) = 1 - e^{-\Gamma/t}$$ (looks like ~~c~~ collision broadened light, but with $-$ sign)



Takes a time $1/\Gamma$ for the atom to get excited, decay again.

distinctly quantum effect.

Example of 2-photon interference, Hong-Ou-Mandel



(note: photon correlation is an intensity (energy) correlation, $g^{(2)}$)

What do you get?

independently

$$|1_{k_2}\rangle \to R_{24}\,|1_{k_4}\rangle + T_{23}\,|1_{k_3}\rangle$$

$$|1_{k_1}\rangle \to R_{13}\,|1_{k_3}\rangle + T_{14}\,|1_{k_4}\rangle$$

energy conservation arguments, must be true.

Energy conservation gives $|R_{24}| = |R_{13}| \equiv R \qquad R^2+T^2=1$

$$|T_{13}| = |T_{14}| \equiv T$$

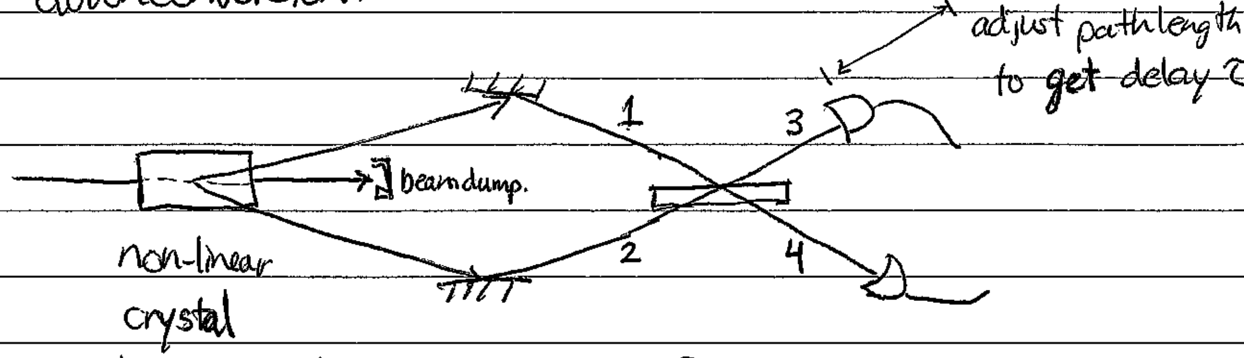
In addition, the phase shifts on reflection have to satisfy:

$$\phi_{13} + \phi_{24} - \phi_{23} - \phi_{14} = \pm\pi$$

must have the correct relative magnitude and phase to interfere and conserve energy.

example: $\left.\begin{array}{ll} R_{13} = R & R_{24} = -R \\ T_{14} = T & T_{23} = T \end{array}\right\}$ what breaks the symmetry?

Experimentally realized via parametric downconversion:



down-converts one photon into 2

$$|1_{k_1}\rangle|1_{k_2}\rangle \to \cancel{R|1_{k_3}\rangle + T|1_{k_4}\rangle}$$
$$= \hat{a}_1^{+}\hat{a}_2^{+}|0\rangle \to \text{create photon in each input arm.}$$

The reflection/transmission properties of the beam splitter are

$$\left.\begin{array}{l} \hat{a}_3^{+} = R\hat{a}_1^{+} + T\hat{a}_2^{+} \\ \hat{a}_4^{+} = -R\hat{a}_2^{+} + T\hat{a}_1^{+} \end{array}\right\} \Rightarrow \begin{array}{l} \hat{a}_1^{+} = R\hat{a}_3^{+} + T\hat{a}_4^{+} \\ \hat{a}_2^{+} = T\hat{a}_3^{+} - R\hat{a}_4^{+} \end{array}$$

$$\hat{a}_1^{+}\hat{a}_2^{+}|0\rangle = (R\hat{a}_3^{+} + T\hat{a}_4^{+})(T\hat{a}_3^{+} - R\hat{a}_4^{+})|0\rangle$$
$$= RT(\hat{a}_3^{+})^2 + T^2\hat{a}_4^{+}\hat{a}_3^{+} - R^2\hat{a}_3^{+}\hat{a}_4^{+} - RT(\hat{a}_4^{+})^2|0\rangle$$

$$= \sqrt{2}RT|2_3 0_4\rangle + (T^2 - R^2)|1_3 1_4\rangle - RT\sqrt{2}|0_3 2_4\rangle$$

for $|T|^2 = |R|^2 = \frac{1}{2}$

$$|\psi_f\rangle = \sqrt{\tfrac{1}{2}}|2_3 0_4\rangle - \sqrt{\tfrac{1}{2}}|0_3 2_4\rangle$$

$$P(2_3 0_4) = |\langle 2_3 0_4|\psi_F\rangle|^2 = \frac{1}{2}|\langle 2_3 0_4|2_3 0_4\rangle|^2 = \frac{1}{2} = P(0_3, 2_4)$$

$P(1,1) = 0$

The photons (completely independent sources) always come out on the same port!

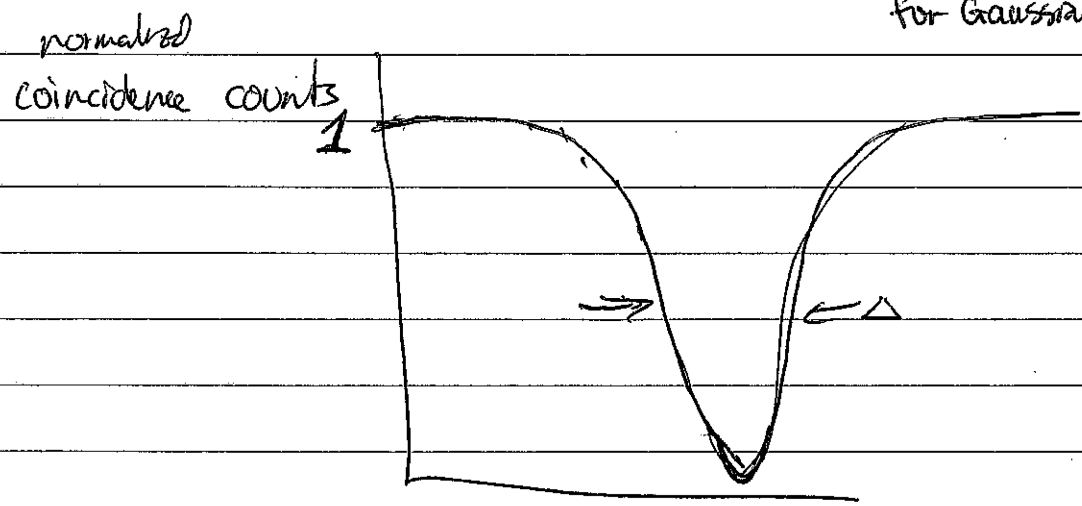
If you factor in a coherence length, ~~or~~ or if the sources are pulsed with a width Δ

$$P(2_3 0_4) = P(0_3 2_4) = 4|R|^2 T^2 (1 + |J|^2) \Rightarrow \tfrac{1}{4}(1+|J|^2)$$

$$P(1_3 1_4) = 1 - 2R^2T^2(1+J^2) \Rightarrow 1 - \tfrac{1}{2}(1+J^2)$$

$$J^2 = \left| \int A_1(t) A_2(t)\, dt \right|^2 \qquad A_t = \text{envelope function.}$$

$\hookrightarrow$ overlap integral $\cdot\, e^{-\frac{1}{2}\Delta^2 (t_0 - t_c)^2}$ for Gaussian pulses.



similar for a correlation width $\rightarrow$ very non-classical.
The classical beam splitter response is Binomial

$P(2_3 0_4) = \frac{1}{4}$ $\quad P(1_3, 1_4) = \frac{1}{4} \times 2$ $\quad P(0_3 2_4) = \frac{1}{4}$