Discussion of photons brings us to quantized light.

How do we move to quantized fields?

\[ E \rightarrow \sum \hat{E}_{\text{m}} \hat{E}_{\text{m}} (\hat{a}_{\text{m}} e^{-i\tilde{k}_{\text{m}} \cdot \tilde{r}} + \hat{a}_{\text{m}}^t e^{i\tilde{k}_{\text{m}} \cdot \tilde{r}}) \]

Note: some authors subsume \( i \) into \( \hat{a} \)

\[ (i\hat{a})^t = -i\hat{a}^t \]

\[ \hat{E} = \sum \hat{E}_{\text{m}} (\hat{a}_{\text{m}} e^{i\tilde{k}_{\text{m}} \cdot \tilde{r}} + \hat{a}_{\text{m}}^t e^{-i\tilde{k}_{\text{m}} \cdot \tilde{r}}) \]

Problem: what is the intensity?

\[ \hat{E}(r) = \hat{E}_{\text{m}} \hat{E}_{\text{m}} \hat{a}_{\text{m}} e^{i\tilde{k}_{\text{m}} \cdot \tilde{r}} \]

\[ \hat{E}^*(r) = \hat{E}_{\text{m}}^* \hat{E}_{\text{m}}^t \hat{a}_{\text{m}}^t e^{-i\tilde{k}_{\text{m}} \cdot \tilde{r}} \]

\[ \frac{1}{2} \varepsilon_0 \varepsilon^2 \| E \| ^2 = c \mathcal{U} = c \sum \tilde{k} \varepsilon_0 \tilde{E}_{\text{m}} (\hat{a}_{\text{m}}^t \hat{a}_{\text{m}} + 1/2) \]

energy density diverges.

Look at detection from a quantum point of view:

use photo-excitation/ionization as a model.

starting in state \( |\tilde{e}_1\rangle \rightarrow 1 \tilde{m} \psi_0 \rangle \), where \( |\psi_0\rangle \) is the state of the detector, we detect light when photons make transitions to \( |\tilde{e}_1\rangle \)

e.g.,

\[ \langle \tilde{e}_1 | \hat{E} | \psi_0 \rangle \rightarrow \text{matrix element} \]

in any event, the radiative part enters independently of the atomic detector part and its contribution to the rate is (for a single mode)

\[ |\langle \tilde{e}_{\text{m}}-1 | \hat{E} | \tilde{e}_{\text{m}} \rangle|^2 \]
Only the destruction operator part contributes,

$$\langle N_{k-1} | \hat{E}_+^+(r) | N_k \rangle^2 = \langle N_k | \hat{E}_-^-(r) | N_{k-2} \rangle \langle N_{k-2} | \hat{E}_-^-(r) | N_k \rangle$$

\(\hat{E}\) is not a vector, it's the magnitude.

For a statistical average, this

The photon contribution to the rate is

$$\sum_{n_{ka}} P(n_{ka}) \langle n_{ka} | \hat{E}^{(0)}_+(r) \hat{E}^{(0)}_-(r) | n_{ka} \rangle = \text{Tr}(\hat{\rho} \hat{E}_+(r) \hat{E}_-(r))$$

(the atomic part drops out as well, if you sum over all possible final detector states, leaving you with an overall quantum efficiency)

This can be generalized to a quantum-optical point by noting

$$\hat{I} = \epsilon_0 c^2 \left( \hat{E}^{(0)}_+(r) \hat{B}^{(0)}_-(r) - \hat{B}^{(0)}_+(r) \hat{E}^{(0)}_-(r) \right)$$

which for polarized parallel light gives

$$\hat{I} = 2 \epsilon_0 c \hat{E}^{(0)}_+(r) \hat{E}^{(0)}_-(r) \frac{\hat{k}}{|\hat{k}|}$$

and

$$\langle \hat{I} \rangle = \frac{\epsilon}{V} \sum_k k w_k \langle n_{ka} \rangle = \frac{\epsilon}{V} \left( \langle E_R \rangle - E_{\text{vac}} \right)$$
Use Heisenberg operators

\[ g^{(1)}(r_1, r_2, z) = \frac{\langle \hat{E}^{(1)}(r_1) \hat{E}^{(1)}(r_2) \rangle}{\sqrt{\langle \hat{E}^{(r_1)} \hat{E}^{(r_2)} \rangle \langle \hat{E}^{+}(r_1) \hat{E}^{+}(r_2) \rangle}} \]

\[ g^{(2)}(r_1, r_2, z) = \frac{\langle \hat{E}^{(1)}(r_1) \hat{E}^{-(r_2)} \hat{E}^{+(r_2)} \hat{E}^{+(r_1)} \rangle}{\langle \hat{E}^{+(r_1)} \hat{E}^{(r_2)} \rangle \langle \hat{E}^{-}(r_2) \hat{E}^{+}(r_2) \rangle} \]

where \[ \langle \hat{O} \rangle = \text{Tr}(\hat{O}) \]

\[ \langle I_{mz} \rangle = \frac{1}{2} e^{i \phi} \frac{1}{4} \left( \langle (\hat{E}(r_1) + \hat{E}(r_2))(\hat{E}^{+(r_1)} + \hat{E}^{+(r_2)}) \rangle \right) \]

One can show \( 0 \leq g^{(2)}(0) \leq \infty \), but one cannot show that \( g^{(2)}(0) \geq 1 \)!

Examples: Start with single mode only, for example.

Note: Single mode is monochromatic.

Classically we saw that

\[ F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \ g^{(1)}(\tau) e^{i\omega \tau} \]

\[ g^{(1)}(\tau) \rightarrow \text{constant}. \]

\[ \phi \] is a sine-wave of fixed phase, pretty boring deterministic light.

\( \phi \) is the only variable freedom.
Scale electric field operator so that
\[ \hat{E}(\phi) = \hat{E}^+(\phi) + \hat{E}^-(\phi) = \frac{1}{2} \hat{a} e^{i\phi} + \frac{1}{2} \hat{a}^+ e^{-i\phi} \]

\[ \hat{x} = \hat{a} + \frac{\hat{a}^+}{\sqrt{2}} \quad \hat{y} = \frac{\hat{a}^+ \hat{a}}{\sqrt{2}} \]

\[ \Rightarrow \hat{E}(\phi_1), \hat{E}(\phi_2) = -\frac{i}{2} \sin(\phi_1 - \phi_2) \]

\[ \Rightarrow \hat{E} \text{ at different } \phi \text{ don't commute.} \]

For single-mode
\[ g^{(1)}(x_1, x_2, z) = \frac{\langle \frac{1}{2} \hat{a} e^{i\phi_1} \hat{a} e^{-i\phi_1} \rangle}{\sqrt{\langle \frac{1}{2} \hat{a}^+ \hat{a} \rangle \langle \frac{1}{2} \hat{a}^+ \hat{a} \rangle}} \]
\[ = e^{i(\phi_2 - \phi_1)} \quad \phi_1 = -kx_1 + \omega t \]
\[ \phi_2 = -kx_2 + \omega(t + z) \]
\[ |g^{(1)}(z)| = 1 \rightarrow \text{kind of expected.} \]

\[ \text{define } S(\hat{E}(\phi)) = \langle \hat{E}(\phi) \rangle \]
\[ N(\hat{E}(\phi)) = (\Delta E(\phi))^2 \quad (\Delta^2 E(\phi))^2 = \langle E^2 \rangle - \langle E \rangle^2 \]
What about \( g^{(2)}(\tau) \)?

\[
g^{(2)}(x, x, \tau) = \frac{1}{\hbar} \frac{\langle \hat{a}^{\dagger} \hat{a} \hat{a}^{\dagger} \hat{a} \hat{a}^{\dagger} \hat{a} \rangle}{\langle \hat{a}^{\dagger} \hat{a} \rangle \langle \hat{a}^{\dagger} \hat{a} \rangle}
\]

\[
= \frac{\langle \hat{a}^{\dagger} \hat{a}^{\dagger} \hat{a} \hat{a} \rangle}{\langle \hat{a}^{\dagger} \hat{a} \rangle^2}
\]

Note: \( g^{(2)} \) doesn't depend on \( \Phi \), (or \( \Delta x \) or \( \Delta \theta \)), but it does depend on the state that is chosen, and it is not always given by deterministic light!

\[
\hat{a} \hat{a} = \hat{a} \hat{a}^{\dagger} - 1
\]

\[
g^{(2)}(\tau) = \frac{n(n-1)}{\langle n \rangle^2} = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2}
\]

\[
= 1 + \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle^2}
\]

Since \( (\Delta n)^2 \geq 0 \), \( (\Delta n)^2 = \langle n^2 \rangle - \langle n \rangle^2 \)

\[
g^{(2)}(\tau) \geq 1 - \frac{1}{\langle n \rangle}
\]

\( g^{(2)}(\tau) \geq 0 \) no matter what.

For finite photon number, this is less than 1! Statistically impossible for classical light.

Compare to EPR!
Specific examples:

number or Fock state:

\[(\Delta n)^2 = 0, \quad g^{(n)} = 1 - \frac{1}{n} \quad \text{for} \quad n \geq 1\]

we showed early in the class, that the quadrature expectation values on number states vanish:

\[\langle n | \hat{X} | n \rangle = \langle n | \hat{Y} | n \rangle = 0\]

but

\[(\Delta \hat{X})^2 = (\Delta \hat{Y})^2 = \frac{1}{2}(n + \frac{1}{2})\]

Coherent states:

\[|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \hat{a} |\alpha\rangle = \alpha |\alpha\rangle \quad \langle \alpha | \hat{a}^\dagger | \alpha \rangle = \langle \alpha | \hat{a} | \alpha \rangle = |\alpha|^2\]

\[\langle \alpha | \beta \rangle \neq 0 = e^{-|\alpha|^2} \rightarrow \text{overcomplete.}\]

\[\langle n \rangle = \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger + \hat{a} | \alpha \rangle = |\alpha|^2\]

\[\langle n^2 \rangle = \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a} | \alpha \rangle = \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger + \hat{a}^\dagger \hat{a} | \alpha \rangle = |\alpha|^4 + 2|\alpha|^2 \langle n \rangle = \langle n \rangle^2 + \langle n \rangle\]

\[(\Delta n)^2 = \langle n \rangle \quad \rightarrow \text{not like a deterministic classical wave.}\]
\[\Delta n = \frac{1}{\langle n \rangle} \text{ relative uncertainty,}\]

not surprising \( P(n) \) is poisson

\[\langle n | \alpha \rangle^2 = \mathcal{E}^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!}\]

Nonetheless,

\[g^{(2)}(\tau) = \frac{\langle a | a^+ a^+ a^+ a^+ | a \rangle}{\langle a | a^+ | a \rangle^2} = 1 + \frac{(\Delta n)^2 - \langle n \rangle}{\langle n \rangle^2} \]

\[= 0\]

\[g^{(2)}(\tau) = 1 \rightarrow \text{no "bunching"} \]

Chaotic light (single mode)

Thermal distribution of photon number (earlier in class)

\[P(n) = \frac{\langle n \rangle^n}{(1+\langle n \rangle)^n} \]

\[(\Delta n)^2 = \langle n \rangle^2 + \langle n \rangle \]

if \((\Delta n)^2 - \langle n \rangle > 0 \rightarrow \text{super poissonian}\)

\[g^{(2)}(\tau) = 2 \text{ for } \langle n \rangle \text{ large.} \]

\[(\Delta n)^2 - \langle n \rangle < 0 \rightarrow \text{sub poissonian}\]
Quadradue.

Squeezed light:

Consider the state $|s\rangle = \hat{S}(s)|0\rangle \rightarrow$ squeezed vacuum

$\hat{S}(s) = e^{\left(\frac{1}{2} s^2 \hat{a}^2 - \frac{1}{2} s \hat{a} \hat{a}^2\right)}$, $s = |s|e^{i\theta}$

$\rightarrow$ looks like a coherent state, but $\hat{a}^2, (\hat{a}^2)$

(immediately indicates needed non-linear processes in matrix, not $E$, but $E^2$

can be shown that $g^{(2)}(\tau) = 3 + \frac{1}{\langle n \rangle}$

$\rightarrow$ super Poisson

$\langle n \rangle \neq 0 = \sinh^2|s|$

$\langle E \rangle$

minimum quantum uncertainty.

Squeezed coherent states.

$g^{(2)}(\tau)$

squeezed $\rightarrow$ squeezed. First real chaotic indication for the need for photons!

(coherent)

$|n\rangle$ (not photoelectric effect)
To deal with multiple modes (e.g., different ports on a beam splitter) or multiple frequencies (pulses of light, e.g.) one needs multimode or continuous mode formalisms.

**Multimode**: add a $\vec{k}$ and $\omega$ dependence,

$$\hat{a}_{\vec{k}, k^\dagger} = \delta_{kk'}, \quad \mid k\rangle = \mid k^\dagger, \vec{k}\rangle \mid \vec{k}\rangle \mid \omega\rangle : \mid \omega\rangle$$

→ get very similar results for “classical” light.

**Continuous mode**: frequency spectrum continuous:
- imagine a box in space (size $L$) with frequency spacing $\Delta \omega = \frac{2\pi c}{L}$
- in the limit that $L \to \infty$, $\Delta \omega \to 0$

\[\Xi_k \to \frac{1}{\Delta \omega} \int d\omega \quad \hat{a}_{kk'} \to \Delta \omega \delta(\omega - \omega')\]

(example for a propagating beam)

\[\hat{a}_{\omega} \to \left[ \hat{a}_{\omega}, \hat{a}_{\omega'} \right] = \delta(\omega - \omega')\]

eg. $\hat{H}_K = \sum_k \hbar \omega_k \left( \hat{a}_k^\dagger \hat{a}_k + \text{vacuum} \right) \to \int d\omega \, \hbar \omega \hat{a}_{\omega}^\dagger \hat{a}_{\omega} + \text{vacuum}$.

\[\hat{H} = \int d\omega \, \hat{a}_{\omega}^\dagger \hat{a}_{\omega}, \quad \text{can define } \hat{a}_{\omega} = \int d\omega' \hat{a}_{\omega'}^\dagger \hat{a}_{\omega'} \cdot \frac{1}{\Delta \omega}\]
Resonance Fluorescence from a single atom.

- shine light (continuously)
  - classically, the radiation is deterministic sine-wave.
  - quantum mechanically: look from the side, anti-bunched, sub poissonian light.

\[ g^{(2)}(t) = 1 - e^{-\frac{1}{\tau} t} \quad \text{(looks like collision broadened light, but with - sign)} \]

\[ g^{(2)}(t) \]

\[ 0 \quad 1 \quad 2 \quad 3 \]

- Takes a time \( \frac{1}{\tau} \) for the atom to get excited, decay again.

- distinctly quantum effect.
Example of 2-photon interference, Hong-Ou-Mandel

\[ |1_{k_2}\rangle \rightarrow R_{24} |1_{k_4}\rangle + T_{23} |1_{k_3}\rangle \]

\[ |1_{k_i}\rangle \rightarrow R_{13} |1_{k_3}\rangle + T_{14} |1_{k_4}\rangle \]

Energy conservation gives \( |R_{24}| = |R_{13}| = R, \quad R^2 + T^2 = 1 \)

\[ |T_{13}| = |T_{14}| = T \]

In addition, the phase shifts on reflection must have to satisfy:

\[ \phi_{12} + \phi_{24} - \phi_{13} - \phi_{23} = \pm \pi \]

Energy conservation arguments must have the correct relative magnitude and phase to interfere and conserve energy.

Example: \( R_{13} = R, \quad R_{24} = -R \) \{ what breaks the \( T_{14} = T, \quad T_{23} = T \) \} symmetry?
Experimentally realized via parametric downconversion:

\[ | 11_{k_1} >, | 11_{k_2} > \rightarrow \mathcal{R} \mathcal{F} \mathcal{H}_k \]

\[ = \hat{a}_1^+ \hat{a}_2^+ | 0 > \rightarrow \text{create photon in each input arm.} \]

The reflection/transmission properties of the beam splitters are

\[ \hat{a}_3^+ = R \hat{a}_3^+ + T \hat{a}_4^+ \]
\[ \hat{a}_4^+ = -R \hat{a}_2^+ + T \hat{a}_1^+ \]

\[ \hat{a}_1^+ \hat{a}_2^+ | 0 > = (R \hat{a}_3^+ + T \hat{a}_4^+)(T \hat{a}_3^+ - R \hat{a}_4^+) | 0 > \]

\[ = RT(\hat{a}_3)^2 + T^2 \hat{a}_1^+ \hat{a}_2^+ - R^2 \hat{a}_3^+ \hat{a}_4^+ - RT(\hat{a}_4)^2 | 0 > \]

for \( | R |^2 = \frac{1}{2} \)

\[ | \gamma_f > = \frac{1}{2} | 12_{3,0} > - \frac{1}{2} | 0, 2 > \]

\[ P(2_{3,0,4}) = | \langle 2_{3,0,4} | \gamma_f > |^2 = \frac{1}{4} | \langle 2_{3,0,4} | 2_{3,0,4} > |^2 = \frac{1}{2} = P(0, 2, 4) \]

\[ D(1, 1) = 0 \]
The photons (completely independent) always come out on the same port.

If you factor in a coherence length, or if the sources are pulsed with a width

\[ P(2, 0_2) = P(0_3, 2_3) = \frac{1}{4} R^2 T^2 (1 + |1|)^2 \Rightarrow \frac{1}{4} (1 + |1|)^2 \]

\[ P(1_3, 1_4) = \frac{1}{4} R^2 T^2 (1 + |1|)^2 \Rightarrow 1 - \frac{1}{4} (1 + |1|)^2 \]

\[ J^2 = \left| \int A_1(t) A_2(t) \, dt \right|^2 \quad A_0 = \text{envelope function.} \]

\[ \Rightarrow \text{overlap integral} \cdot e^{-\frac{1}{2}(t-t_0)^2} \quad \text{for Gaussian pulses.} \]

normalized coincidence counts

\[ P(2, 0_2) = \frac{1}{4} \quad P(1_3, 1_4) = \frac{1}{4} \times 2 \quad P(0_3, 2_3) = \frac{1}{4} \]

Similar for a correlation width \( \Rightarrow \) very non-classical.

The classical beam splitter response is Binomial