Define a second moment to be a correlation of intensities or a four-field correlation

\[
g^{(2)}(\tau) = \frac{\langle \overline{I}(t) \overline{I}(t+\tau) \rangle}{\langle \overline{I}^2 \rangle} = \frac{\langle E^*(t) E^*(t+\tau) E(t+2) E(t) \rangle}{\langle E^*(t) E(t) \rangle^2}
\]

If stationary, \( g^{(2)}(-\tau) = g^{(2)}(\tau)^* \)

Cauchy-Schwarz inequality imply

\[
2 \overline{I}(t) \overline{I}(t+\tau) \leq \overline{I}(t)^2 + \overline{I}(t+\tau)^2
\]

\[
\langle (I(t)^2 - 2I(t)I(t+\tau))^2 \rangle \geq 0
\]

\[
\langle I(t+\tau)^2 \rangle - 2 \langle I(t)I(t+\tau) \rangle \geq 0
\]

for stationary light,

\[
2 \langle I(t)^2 \rangle - 2 \langle I(t)I(t+\tau) \rangle \geq 0
\]

\[
\langle I(t)^2 \rangle \geq \langle I(t)I(t+\tau) \rangle
\]

\[
g^{(2)}(0) \geq g^{(2)}(\tau)
\]

Similar arguments can show \( g^{(2)}(0) \geq 1 \)

\[
g^{(2)}(\tau) \geq 0, \tau \neq 0
\]
Examples: Doppler or collision broadened light. (anything that is a sum over independent radiances)

\[
\langle E^*(i) E^*(i+2) E(i+2) E(i) \rangle = \frac{\sum_{\text{box}} \langle I_i(4) I_{i+2}(4) \rangle}{\sum_{\text{box}} \langle I_i(4) I_{i+2}(4) \rangle}
\]

\[
= \sum_{\text{box}} \langle I_i(4) I_{i+2}(4) \rangle + \sum_{i,j} \langle I_i(4) I_j(4) \rangle + \sum_{i,j} \langle E^*(i) E^*(i+2) E(i+2) E_j(4) \rangle
\]

dominated by these large pairwise sums.

assume stationary,

\[
\langle E^* \rangle = N_{\text{atoms}} \left( \langle I_i(4) \rangle^2 + \langle E^*(i) E(i+2) \rangle \right)^2
\]

\[
g^{(2)}(2) = 1 + |g^{(1)}(0)|^2 \quad \text{(assumes very large number of atoms contribute)}
\]

The wave:

\[
g^{(2)}(2) = 1 \quad \text{– intensity intensity correlation}
\]
Note: even for filtered thermal or collision broadened light, measured on time scales \( t \ll \tau_c \), there is a difference from deterministic sine-wave light.

\[
g^{(1)}(\tau) = 1 \quad \text{thermal}
\]

\[
g^{(2)}(\tau) \ll \tau_c = 2 \quad \text{thermal}
\]

\[
g^{(2)}(\tau) \gg \tau_c = 1 \quad \rightarrow \text{sine wave}
\]

\[
g^{(1)} \quad \rightarrow \text{pure sine}
\]

\[
g^{(2)} \quad \rightarrow \text{thermal}
\]

Excess fluctuations always present for incoherent summed light!
Example astronomical applications of \( q^1 \) and \( q^2 \):

- Starlight transverse coherence:

  \[ q^\ast \rightarrow R \rightarrow \delta \text{Earth} \]

  *Take \( R = R \ast \) to be the size of the Earth (By double slit interferometer)*

  Standard star \( \ast \)

  Angular \( \Delta \ast = \frac{1}{10^9 \text{m}} \)

  \[ \begin{align*}
    \text{Transverse coherence length} &\approx R \left( \frac{\lambda}{d} \right) \\
    &\approx 1 \text{m}
  \end{align*} \]

  Michelson stellar interferometer:

  \[ \begin{align*}
    I \propto \langle |E|^2 \rangle &= \left| E_k^* (e^{ik \cdot r_1 + i\phi} + e^{ik' \cdot r_2}) + E_k' (e^{ik' \cdot r_2} + e^{ik \cdot r_1}) \right|^2 \\
    &\propto 2|E_k|^2 + |E_k'|^2 + |E_k|^2 (e^{i(k-k') \cdot r_1} + e^{i(k-k') \cdot r_2}) \\
    &\quad + |E_k'|^2 (e^{i(k-k') \cdot r_2} - e^{i(k-k') \cdot r_1})
  \end{align*} \]

  (Assumed light is filtered so \( \nu_k = \nu'_k \), so no relative time dependence.)
Assuming thermal light. $\langle E_k \rangle = 0$

$\langle E_k^* E_{k'} \rangle = 0$ (uncorrelated sources)

$I \propto 2I_0 \left( 2 + \cos(k \cdot \vec{r}_1 - \vec{r}_2) + \cos(k' \cdot \vec{r}_1' - \vec{r}_2') \right)$

$= 4I_0 \left( 1 + \cos(k + k') \cdot \frac{\vec{r}_1 - \vec{r}_2}{2} \cdot \cos(\vec{k} - \vec{k}) \cdot \frac{\vec{r}_1' - \vec{r}_2'}{2} \right)$

$= 4I_0 \left( 1 + \cos(k + k') \cdot \frac{\vec{r}_1 - \vec{r}_2}{2} \right) \cos(\vec{k} - \vec{k}) \cdot \frac{\vec{r}_1' - \vec{r}_2'}{2}$

$\Delta k \approx \Delta k' \quad I = 4I_0 \left( 1 + \cos(k + k') \cdot \frac{\vec{r}_1 - \vec{r}_2}{2} \right) \cos(\frac{\pi r_0 \Phi}{\lambda}) \Delta r_0 >> \lambda$

$\rightarrow$ can be used to measure the beating of the two interference against each other $\rightarrow$ get the angle

There is a drawback: $\hat{k} + \hat{k}'$ is a high fringe term, very sensitive to atmospheric turbulence.

Hanbury Brown - Twiss.

merely place two photodetectors separated by $r_0$

$$I = |E_k e^{i(k \cdot \vec{r})}|^2 + |E_k' e^{i(k' \cdot \vec{r}')}|^2$$

$$I = |E_k|^2 + |E_k'|^2 + E_k E_k' e^{i(k \cdot \vec{r} + k' \cdot \vec{r}')} + \text{c.c.}$$
\[ < I(r,\hat{t}) I(r',\hat{t}) > = \left< \left| E_k \right|^2 + \left| E_{k'} \right|^2 + E_k E_{k'} \frac{\epsilon^{-i(k \cdot r - k' \cdot r')}}{\epsilon^{i(k \cdot r - k' \cdot r')}} \right> \]

\[ \times \left[ \left| E_k \right|^2 + \left| E_{k'} \right|^2 + E_k E_{k'} \left( \frac{\epsilon^{-i(k \cdot \bar{r}_Z - k' \cdot \bar{r}_Z)}}{\epsilon^{i(k \cdot \bar{r}_Z - k' \cdot \bar{r}_Z)}} + \text{c.c.} \right) \right> \]

\[ = \left< \left( |E_k|^2 + |E_{k'}|^2 \right)^2 + |E_k|^2 |E_{k'}|^2 \left( e^{i(k \cdot \bar{r}_Z - k' \cdot \bar{r}_Z)} + \text{c.c.} \right) \right> \]

Terms that average to zero have been dropped.

\[ = \text{I}_{\text{E},\text{E}} g^{(2)}(\bar{r}_1,\bar{r}_2,0) \]

\[ < I(r,\hat{t}) I(r,\hat{t}) > = \left< \left( I_1 + I_2 \right)^2 \right> + \text{I}_1 \text{I}_2 \cos(k \cdot \bar{r}_Z) \]

\text{Interference! and no } k + k' \text{ term!}

\text{This caused a stir: multiplication is in the photo current electronics. How can they "interfere"?}

\text{Noise - correlation interferences.}

\[ S \]

\[ \Delta \Theta_1 \sim \text{random} \]

\[ \Delta \Theta_2 \sim \text{random} \]

\[ \Delta \Theta_2 \sim \text{correlated to } \Delta \Theta_1 \]

\[ \text{by } (k \cdot \bar{r}_Z) r_0 \]

\[ S \]

\[ S \]

\[ S' \]

\[ S' \]

\[ S \]

\[ \text{When } I_1 \text{ is large, } I_2 \text{ is large also, vice versa. } (k \cdot \bar{r}_Z) r_0 = \pi \]

\[ \text{will be true for any amount of } g^{(2)} \]
How do we deal with quantum light?

replace $E$ $\rightarrow$ $E - \sum_k \hat{f}_k \delta (k)$

Effects of detector bandwidth: the original H-B-T experiment used light from mercury lamp at 435 nm and used a 50-50 beam splitter.

For ideal detectors (infinite bandwidth) this will be

$$\langle \hat{I}_a (t) \hat{I}_b (t+\tau) \rangle = \frac{1}{4} \langle \hat{I}_a (t) \hat{I}_a (t+\tau) \rangle$$

$$= \frac{1}{4} \sum_{\tau} \hat{I}_b \hat{I}_b \hat{g}^{(2)} (\tau)$$

(actually the experiment is set up to subtract the average)

$$\propto \hat{g}^{(2)} (\tau) - 1$$

however, if the detectors have finite bandwidth (as the first ones surely did for H-B-T) must aver age.

e.g. Collisionally broadened light

$$\hat{g}^{(2)} (\tau) = 1 = |g^{(1)}(\tau)|^2 = e^{-2|\tau|/\tau_c}$$

$$\frac{\langle \hat{g}^{(2)} (\tau) - 1 \rangle^W}{\langle \hat{I} \rangle^W} = \frac{1}{T^2} \int_0^T \int_0^T e^{-2|t_a - t_b|/\tau_c} T \rightarrow \infty \rightarrow 0 \hat{g}^{(2)} (\tau)$$

for H-B-T

$$\tau / T \approx 10^{-5} \approx \frac{T_c}{2 T^2} \left( e^{-\frac{2 T_c}{T}} - 1 + \frac{2 T}{T_c} \right) \quad T \gg T_c \rightarrow \frac{T_c}{T} \text{ down by the ratio.}$$
Look at time-averaged moments of the intensity,

\[ \langle I^2(t) \rangle \rightarrow \text{average intensity}, \sum_i I_i \rightarrow \text{intensity from each independent source.} \]

\[ \langle I^2(t) \rangle \rightarrow 2\langle I \rangle^2 \rightarrow \text{true for all random sources of light 'chaotic', consequence of assuming collections of dimension random and phase.} \]

\[ \Delta I^2 = \langle I^2 \rangle - \langle I \rangle^2 = \langle I \rangle^2 \rightarrow \text{fluctuations equal the average value.} \]

(remember the thermal distribution of photons?)

\[ \overline{n}^2 - \overline{n}^2 = \overline{n}^2 \]

most likely value is 0, somewhat counter intuitive.

Note this is not a measure of \( \sigma_I^2 \),

this is an equal time measure of \( I \).
Intensity fluctuations of chaotic light = probability distribution.

\[
\frac{\left< \left( I(t) - \left< I(t) \right> \right)^2 \right>}{\left< I^2 \right>} = \frac{\left< I^2(t) \right> - \left< I \right>^2}{\left< I \right>^2} = g^2(0) - 1 = 1 \text{ for chaotic light (large number of random sources)}
\]

= 0 for pure sine wave.

What is the probability distribution of fields for chaotic light?

\[E_0 a(t)\text{ is total field of } N \text{ radiating sources with random phases.}\]

For \(N\) large, this is a 2D random walk problem, with distribution

\[p[a(t)] = \frac{1}{\pi N} e^{-\left( g^2(t)/N \right)}\]

Gaussian distribution of field amplitude.

(not to be confused with the Gaussian or Lorentzian time correlation dependence for different kinds of chaotic light) this is the probability distribution averaged over all time of \(E\) value.)
$E_0(t) = E_0 e^{it \alpha(t)}$ is the amplitude (not $e^{i \omega t}$ time dependence.)

$\alpha$ is complex.

$\Re [E_0(t)] = \cos \text{ quadrature}$

$\Im [E_0(t)] = \sin \text{ quadrature}$

$|E_0(t)|$ is the magnitude of field

$p[|E_0(t)|] \rightarrow \text{gaussian with most probable value 0}$

$\rightarrow N$ randomly phased radiators give an average

$(\Delta |E_0(t)|^2) = \frac{1}{2} E_0^2 N$ field of $\sqrt{N} E_0$

$N$ in-phase radiators give $NE_0$

$I = \frac{1}{2} \varepsilon_0 c |E(t)|^2$, $p[I]$ is $\frac{1}{I}$ exponential.

again, the most probable value is $I(t) = 0$!

Recall that a thermal distribution of photons looks similar