Classical Theory of Coherence:

First: "There is no general agreement on the precise meaning of the term coherence."

Usually "coherence" is used to describe light that shows some kind of stable interference in an experimental set up.

We will try to describe "coherence" properties in terms of correlation functions.

Given a point in space where a light field is passing, how can we characterize this field? (for now, look only at the time dependence of the field at a single point, so-called longitudinal coherence, ignore transverse directions.) (Also ignore polarization for now.)

All light comes from material sources: convenient to consider source when describing idealized source light

How can we characterize this EM field? It will be convenient to use statistical argument.
- We could specify $E(t)$ in its entirety.
- But there aren't any convenient $E(t)$ measurement devices in optical regime.
- Requires lots of detailed information that may not be physically relevant.
- Ignores the fact that light from real sources may have statistical properties that simplify discussion.

We could specify the Fourier transform in its entirety. $\Rightarrow \hat{E}(\omega)$

$\Rightarrow$ similar measurement difficulties need phase information.

Detectors measure intensity: $I \propto |E|^2$, usually averaged over some bandwidth, $I \propto \langle |E|^2 \rangle_{\tau}$.
$\Rightarrow$ necessarily throws away some information.

Specify $I(t)$ $\Rightarrow$ detector bandwidth not large enough to include optical frequencies, $\Rightarrow$ can be done for lots of sources. $\Rightarrow$ provides more information.

Specify $\hat{I}(\omega)$, this can be done to some degree, e.g. using optical spectrum analyzers, or FFTing $I(t)$.

$\Rightarrow$ Note: any time dependence other than sine or cosine implies spectral width to $\hat{E}(\omega), \hat{I}(\omega)$.
Look at the different types of light.

Naturally averaged: $\langle I(t) \rangle$ over some time scale (detector bandwidth $\Delta$), always much slower than $\omega_c = \frac{\hbar}{\Delta}$.

Example of $I(t)$ for different sources:

Chaotic light—light from a collection of independent sources.

Light scattered from atoms that collide.

Example: collisional broadening: elastic collisions are very brief, discontinuously adjust the phase.

$$\rightarrow \text{for single atom}$$

Discontinuous phase jumps occur with probability $p(t) = \frac{1}{\tau_0} e^{-(t/\tau_0)}$, $\tau_0 = 4 \sigma_{col} N V_{th}$.

Typically $\omega_c \tau_0 \approx 10^5$.

$N$ is density $V_{th} = \sqrt{\frac{2kT}{m}}$.

$$E(t) = \sum_{\text{atoms}} E_i(t)$$

$$= E_0 e^{-i\omega_c t} \sum E_i e^{i\phi_i} = E_0 e^{-i\omega_c t} a(t) e^{i\phi}$$

$I$ is cycle averaged intensity.

random walk in phase space.
\[ I(\omega) \approx \frac{\Gamma_4 \Gamma_{col}}{\delta^2 + \frac{\Gamma_4 \Gamma_{col}}{\delta} \left( 1 + \frac{\Gamma_{col}}{\Gamma_{iso}} \right)} \]

\[ \Gamma_{col} = \frac{1}{\Gamma_{iso}} \]

- Doppler broadening.

\[ E(t) = \sum E_0 e^{i \omega_i t} \rightarrow \omega_i \rightarrow \text{thermal distribution} \]

- As we know from homework:
  - gives rise to gaussian frequency + small wings - less fine components \( \rightarrow \) smoother.

Pure sine wave (classically possible, n.b. not possible QM.)

\[ I(t) \]

\[ FT \rightarrow \delta \text{ function} \]
In practice, \( I(t) \) fluctuates more rapidly than can easily be detected, plus, we want to know about coherence + interference.

Use an interferometer to self-reference the beam with a delay \( t_1 = t - \frac{d_1}{c} \), \( t_2 = t - \frac{d_2}{c} \)

\[
\text{E}_{\text{out}} = RT \ E(t_1) + TR \ E(t_2) \rightarrow \text{path length delay}
\]

\[
\text{I}_{\text{out}} = \frac{1}{2} \varepsilon_0 c R^2 T^2 \left( |E(t_1)|^2 + |E(t_2)|^2 + 2 \text{Re}(E^*(t_1) E(t_2)) \right) \]

average over bandwidth of the detector: \( \text{assuming } R^2 T^2 = \frac{1}{2} (\Delta f)^2 \)

\[
\langle \text{I}_{\text{out}} \rangle = \frac{\langle I(t_1) \rangle + \langle I(t_2) \rangle + \varepsilon_0 c R \text{Re}(E^*(t_1) E(t_2))}{4}
\]

\( \langle I(t) \rangle \) is just the intensity averaged if there were no interferometer. In the limit of infinite averaging time \( \rightarrow I_0 \)

\[
\langle E^*(t_1) E(t_2) \rangle = \frac{1}{T} \int_T dt_1 E^*(t_1) E(t_2) \text{ for stationary sources}
\]

\[
\langle E^*(t) E(t+\tau) \rangle = \frac{1}{T} \int_T dt E^*(t) E(t+\tau)
\]

\( \tau \gg \text{correlation time} \), \( T \ll \text{drift time} \) → good mean
nice measurement, because even in the limit of infinitely slow detectors, we get a correlation measurement out: statistical average over light properties that differ in time by \( t = \frac{1}{c}(l_2 - l_1) \)

\[ g^{(1)}(2) = \frac{\langle E^*(t) E(t+2) \rangle}{\langle I \rangle} \]

0 \leq g^{(1)}(2) \leq 1

For stationary light (independent of \( t \))

\[ g^{(1)}(-2) = \frac{\langle E^*(t) E(t-2) \rangle}{\langle I \rangle} = \frac{\langle E^*(t+2) E(t) \rangle}{\langle I \rangle} = g^{(1)}(2)^* \]

real part of \( g^{(1)} \) is the same at \( +2 \pm 2 \)

\[ \frac{1}{2} \langle I_{\text{out}} \rangle = \frac{1}{2} \langle I(t) \rangle (1 + \text{Re}[g^{(1)}(2)]) \]

examples:

collision broadened:

\[ \langle E^*(t) E(t+2) \rangle = E_0^2 e^{-i \omega_0 2} \langle e^{i \phi(t)} e^{i \phi(t+2)} \rangle \]

coherently random phase after each collision

\[ = E_0^2 e^{-i \omega_0 2 + \frac{\sigma^2}{2} \text{coh}^2} \]

probability that \( \phi(t) + \phi(t+2) = 0 \)

uncorrelated is

\[ g^{(1)}(2) = e^{-i \omega_0 2 - \frac{\sigma^2}{2}} \]

\[ 1 - \int_0^\infty p(t) dt \]
\[ |g^{(1)}(\tau)| \]

Tells over what range interference fringes can be seen.

\[ T_{\text{coh}} \sim \]

\[ \tau_{\text{coh}} \]

\[ \frac{\Delta^2}{2\Delta} \]

Example:

For Doppler broadened source:

\[ \langle E^*(t) E(t+\tau) \rangle = \sum_{\text{channels}} E_0 \, e^{-i\omega_0 \tau} \]

averaged all other cross terms to zero.

\[ \rho_{ii} \]

is averaged over a Doppler spectrum.

\[ g^{(1)}(\tau) = e^{-i\omega_0 \tau - \frac{1}{2} \Delta^2 \tau^2} \]

\[ \Delta = \text{width of Doppler spectrum} \]

Coherence time can be defined as \( T_{\text{coh}} = \frac{\pi}{\Delta} \)

Example:

Sin wave \( g^{(1)}(\tau) = e^{-i\omega_0 \tau} \)

\[ |g^{(1)}(\tau)| \]

\[ \rho_{ii} \]

\[ \text{actual fringes will be} \]

\[ \text{coherence time can be defined as} \]

\[ T_{\text{coh}} = \frac{\pi}{\Delta} \]

\[ \text{example:} \]

Sin wave \( g^{(1)}(\tau) = e^{-i\omega_0 \tau} \)}
There is a connection between 1st order correlation function and the frequency spectrum:

\[ \hat{E}_x(\omega) = \int_{-\Delta t}^{\Delta t} E(t) \sin(\omega t) \, dt \]

Power spectrum

\[ \hat{f}(\omega) = \left| \frac{E_x(\omega)}{T} \right|^2 = \frac{1}{2\pi T} \int_{-\Delta t}^{\Delta t} dt \int_{-\Delta t}^{\Delta t} E^*(t) E(t') e^{-i\omega(t-t')} \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left< E^*(t) E(t+\tau) \right> e^{i\omega\tau} \]

\[ F(\omega) = \hat{f}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \left< g^{(1)}(\tau) \right> e^{i\omega\tau} \]

\[ g^{(1)}(r_1, \vec{r}_2, \tau) = \left< E^*(\vec{r}_1, t) E(\vec{r}_2, t+\tau) \right> \]

\[ \sqrt{\left< |E(\vec{r}_2, t+\tau)|^2 \right> \left< |E(\vec{r}_1, t)|^2 \right>} \]

\[ \Rightarrow \text{independent of } \tau \]

\[ \Rightarrow \text{double-slit diffraction and Michelson interferometry are the same.} \]
we can define a coherence length

$\tau_{\text{coh}}, \tau_{\text{depin}}$ - coherence time

$l = C \tau_{\text{coh}}$ - coherence length

Transverse coherence, e.g. from a star

when will you get interference? assume equal time measurement to see purely geometric averaging effects $g^{(0)}(\hat{r}_1, \hat{r}_2, z)$

$\tau = \frac{||\hat{r}_1 - \hat{r}_2||}{c}$ equivalent to

when $a \Delta \theta \leq \lambda$. Choose $\Delta \theta$ exactly at coherence width

So the area of coherence is $(R \Delta \theta)^2 \sim R^2 \left(\frac{\lambda}{a}\right)^2$

Coherence volume $\Delta V = \frac{4 \pi}{a}$

coherence area scales $\frac{R^2}{A^2} \sim \frac{\lambda^2}{\lambda^2 - \lambda^2}$

like $R^2 \rightarrow$ further away, better interference, emittance light.
one can approximate deterministic light with a laser. one can also try to filter thermal light to get coherence - hard to do and you lose power.

Examples:

- best thermal light: $\Delta \nu \sim 10^9 \text{ Hz}$
- laser light: $\Delta \nu \sim 0.1 \text{ Hz}$

Coherence time: thermal $10^3 \text{ sec}$

Coherence length: $3 \text{ m}$

$3 \times 10^9 \text{ m}$

First order correlation gives a lot of information about the coherence properties of a source, but does not specify everything! Statistical information.

What about correlations of intensities? (like shown in figures of $I(|\psi\rangle)$)

- consider chaotic light.