density matrix treatment of the interaction of a 2-level atom with a monochromatic E-M field.

$\mathcal{H}_0$ is the atom Hamiltonian

$\mathcal{H}_1 = -\vec{d}\cdot\vec{E}$ is the interaction

for now we treat the applied field

$E(t) = E\cos\omega t$ as classical and ignore the vector character (we return to the vector character when we study angular momentum)

all of the Q.M. operator nature of $\mathcal{H}_1$ is in $\hat{d}$

for simplicity we take $\langle e|d|g\rangle = \langle g|d|e\rangle = d_{eg} = d_{ge}$

and $\langle e|d|e\rangle = \langle g|d|g\rangle = 0$

also $\langle g|\mathcal{H}_0|g\rangle = 0 \qquad \langle e|\mathcal{H}_0|e\rangle = \hbar\omega_0$

$\langle g|\mathcal{H}_0|e\rangle = \langle e|\mathcal{H}_0|g\rangle = 0$ (obviously)

we can write $\mathcal{H}_0$ and $\mathcal{H}_1$ as

$$\mathcal{H}_0 = \hbar\omega|e\rangle\langle e|$$

$$\mathcal{H}_1 = -\hat{d}\,E(t) = -E\,d_{eg}\left(|e\rangle\langle g| + |g\rangle\langle e|\right)\cos\omega t$$

because these give the right matrix elements

Describing a particular photon state classically.

Take a photon to be in a classical state $|\alpha\rangle$, which is a coherent state

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n e^{-\frac{|\alpha|^2}{2}}}{\sqrt{n!}} |n\rangle \qquad |\alpha|^2 = \bar{n} \gg 1$$

avg. # of photons

(all photons are in the same mode, so drop $k, \lambda$ subscripts)

To calculate coherent matrix elements (ignore spontaneous emission for the moment) use the Heisenberg picture (because we want the time dependence to be in the electric field for classical picture)

$$\vec{E} = \mathcal{E}_0 \vec{\epsilon} \left( \hat{a} e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t} \right) \qquad \left(\text{take } \vec{R} = 0 \text{ in } e^{i\vec{k}\cdot\vec{R}}\right)$$

when considering matrix elements of $\hat{\vec{d}} \cdot \hat{\vec{E}}$ between states $|\text{atom}\rangle|\alpha\rangle$, $|\text{atom}'\rangle|\alpha'\rangle$, one will have the elements.

$\alpha$ is complex

Using $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

$$\langle \alpha' | \hat{a} | \alpha \rangle = \alpha \langle \alpha' | \alpha \rangle$$

$$= \alpha \underbrace{e^{-|\alpha'-\alpha|^2/2} e^{(\alpha'^*\alpha - \alpha'\alpha^*)}}$$

$$\langle \alpha' | \hat{a}^\dagger | \alpha \rangle = \alpha'^* \quad ''$$

note that if $|\alpha|^2 = \bar{n} \gg 1$, then the coherent states are almost orthogonal, in number and phase, so $\langle \alpha' | \alpha \rangle \cong \delta^2(\alpha' - \alpha)$ (where $\delta^2$ means $\delta$-function in complex space) so you pretty much only couple to $\langle \alpha |$:

$$\langle \alpha | \hat{\vec{E}} | \alpha \rangle = \mathcal{E}_0 \vec{\epsilon} \left( \alpha e^{i\omega t} - \alpha^* e^{-i\omega t} \right) \Big\} \text{ classical, complex \#'s}$$

parameterize the peak classical electric field by $\vec{E}$, choose a phase

$$\langle \alpha | \hat{\vec{E}} | \alpha \rangle = \vec{E} \cos(\omega t) \rightarrow \text{just what you'd expect.}$$

Using the von Neuman eqn:

$$\dot{\sigma} = \frac{-i}{\hbar}\left[\mathcal{H}_0 + \mathcal{H}_1, \sigma\right]$$

define $-dE = \hbar\Omega$
(and hope we did it right so $\Omega$ is indeed the Rabi freq.)

so
$$\mathcal{H}_0 = \hbar\omega |e\rangle\langle e|$$
$$\mathcal{H}_1 = \hbar\Omega\left(|e\rangle\langle g| + |g\rangle\langle e|\right)\cos\omega t$$

and we calculate each element of $\dot{\sigma}_{ij}$

$$\dot{\sigma}_{ee} = \frac{-i}{\hbar}\left\{\langle e|(\mathcal{H}_0+\mathcal{H}_1)\sigma|e\rangle - \langle e|\sigma(\mathcal{H}_0+\mathcal{H}_1)|e\rangle\right\}$$

$$-i\left\{\langle e|\left(\omega_0|e\rangle\langle e| + \Omega(|e\rangle\langle g| + |g\rangle\langle e|)\cos\omega t\right)\sigma|e\rangle\right.$$
$$\left. - \langle e|\sigma(\omega_0|e\rangle\langle e| + \Omega|e\rangle\langle g| + |g\rangle\langle e|\cos\omega t|e\rangle\right\}$$

$$= -i\left\{\omega_0\sigma_{ee} + \Omega\sigma_{ge}\cos\omega t - \omega_0\sigma_{ee} - \Omega\sigma_{eg}\cos\omega t\right\}$$

$$\dot{\sigma}_{ee} = i\Omega\left(\sigma_{eg} - \sigma_{ge}\right)\cos\omega t$$

similarly (or by conservation of pop.)

$$\dot{\sigma}_{gg} = -i\Omega\left(\sigma_{eg} - \sigma_{ge}\right)\cos\omega t$$

$$\dot{\sigma}_{ge} = -i\left\{\langle g|\left(\omega_0 |e\rangle\langle e| + \Omega\left(|e\rangle\langle g| + |g\rangle\langle e|\right)\cos\omega t\right)\sigma|e\rangle\right.$$

$$\left. - \langle g|\sigma\left(\omega_0|e\rangle\langle e| + \Omega\left(|e\rangle\langle g| + |g\rangle\langle e|\cos\omega t\right)|e\rangle\right\}$$

$$= -i\left\{ 0 + \Omega\sigma_{ee}\cos\omega t - \omega_0\sigma_{ge} - \Omega\sigma_{gg}\cos\omega t\right\}$$

$$\dot{\sigma}_{ge} = i\omega_0\sigma_{ge} - i\Omega\left(\sigma_{ee} - \sigma_{gg}\right)\cos\omega t$$

and similarly:

$$\dot{\sigma}_{eg} = -i\omega_0\sigma_{eg} + i\Omega\left(\sigma_{ee} - \sigma_{gg}\right)\cos\omega t$$

these equations are exactly equivalent to the equations already derived for the state amplitudes (although they will be more convenient for connecting to the vector model)

but, these density matrix equations allow us to take account of spontaneous emission, something we cannot do easily will wavefunctions and state amplitudes.

to account for spontaneous emission we write

$$\dot{\sigma}_{ee\,(\text{spont})} = -\Gamma\,\sigma_{ee}$$

where $\Gamma$ was calculated in the quantized field formalism

and we write $\dot{\sigma}_{gg} = +\Gamma\,\sigma_{ee}$

this takes care of the populations $\sigma_{gg}$ and $\sigma_{ee}$

what about the "coherences" $\sigma_{ge}$, $\sigma_{eg}$ ?

recall that for the pure state $\sigma_{eg} = c_e c_g^*$

in free evolution $c_e(t) = c_e(0)\,e^{-i\omega_0 t}$

if we add spont. emission:

$$c_e(t) = c_e(0)\,e^{-i\omega_0 t}\,e^{-\Gamma t/2}$$

$$\text{or,}\quad \omega_0 \rightarrow (\omega_0 - i\Gamma/2)$$

so, we can write

$$\dot{\sigma}_{ge\,(\text{spont})} = -\frac{\Gamma}{2}\sigma_{ge}$$

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Now, we add all these terms to get the full equations of motion.

$$\dot{\sigma}_{ee} = i\Omega\,(\sigma_{eg} - \sigma_{ge})\cos\omega t - \Gamma\sigma_{ee}$$

$$\dot{\sigma}_{gg} = -i\Omega\,(\sigma_{eg} - \sigma_{ge})\cos\omega t + \Gamma\sigma_{ee}$$

$$\dot{\sigma}_{ge} = i\omega_0\,\sigma_{ge} + i\Omega\,(\sigma_{ee} - \sigma_{gg})\cos\omega t - \frac{\Gamma}{2}\sigma_{ge}$$

$$\dot{\sigma}_{eg} = -i\omega_0\,\sigma_{eg} + i\Omega\,(\sigma_{ee} - \sigma_{gg})\cos\omega t - \frac{\Gamma}{2}\sigma_{eg}$$

recall that we used a perturbation $\mathcal{H}_1$ of the form:

$$\hbar\Omega\left[|g\rangle\langle e| + |e\rangle\langle g|\right]\cos\omega t =$$

$$\frac{\hbar\Omega}{2}\left[|g\rangle\langle e| + |e\rangle\langle g|\right]\left(e^{i\omega t} + e^{-i\omega t}\right) =$$

$$\frac{\hbar\Omega}{2}\left(|g\rangle\langle e|e^{i\omega t} + |e\rangle\langle g|e^{-i\omega t} + |g\rangle\langle e|e^{-i\omega t} + |e\rangle\langle g|e^{i\omega t}\right)$$

each of these terms corresponds to going from ground to excited or excited to ground, with absorption or emission of a photon. Two are "resonant" that is ground → excited with absorption

$|e\rangle\langle g| e^{-i\omega t}$

and excited to ground with emission

$|g\rangle\langle e| e^{i\omega t}$

and two are "anti-resonant", that is

ground → excited with emission

$|e\rangle\langle g| e^{i\omega t}$

and excited to ground with absorption

$|g\rangle\langle e| e^{-i\omega t}$

recall that in the quantized field formalism

$$\vec{\hat{A}} \sim \hat{a}\, e^{i(k\cdot r - \omega t)} + \hat{a}^{+} e^{-i(k\cdot r - \omega t)}$$

"negative" frequency went with anihilation (absorption)

while "positive" frequency went will creation (emission)

this is an alternate interpretation of the non-resonant terms as involving the "wrong" absorption/emission process as before, we will drop the non-resonant terms

so, the equations:

$$\dot{\sigma}_{ee} = i\Omega\,(\sigma_{eg} - \sigma_{ge})\cos\omega t - \Gamma\sigma_{ee}$$
$$\dot{\sigma}_{gg} = -i\Omega\,(\sigma_{eg} - \sigma_{ge})\cos\omega t + \Gamma\sigma_{ee}$$
$$\dot{\sigma}_{ge} = i\omega_0\,\sigma_{ge} - i\Omega\,(\sigma_{ee} - \sigma_{gg})\cos\omega t - \Gamma/2\,\sigma_{ge}$$
$$\dot{\sigma}_{eg} = -i\omega_0\,\sigma_{eg} + i\Omega\,(\sigma_{ee} - \sigma_{gg})\cos\omega t - \Gamma/2\,\sigma_{eg}$$

become, with only resonant terms:

$$\dot{\sigma}_{ee} = \frac{i\Omega}{2}\left(\sigma_{eg}\,e^{i\omega t} - \sigma_{ge}\,e^{-i\omega t}\right) - \Gamma\sigma_{ee}$$

$$\dot{\sigma}_{gg} = \frac{-i\Omega}{2}\left(\sigma_{eg}\,e^{i\omega t} - \sigma_{ge}\,e^{-i\omega t}\right) + \Gamma\sigma_{ee}$$

$$\dot{\sigma}_{ge} = i\omega_0\,\sigma_{ge} - \frac{i\Omega}{2}(\sigma_{ee} - \sigma_{gg})\,e^{i\omega t} - \frac{\Gamma}{2}\sigma_{ge}$$

$$\dot{\sigma}_{eg} = -i\omega_0\,\sigma_{eg} + \frac{i\Omega}{2}(\sigma_{ee} - \sigma_{gg})\,e^{-i\omega t} - \frac{\Gamma}{2}\sigma_{eg}$$

Just as in the solution of Schrod. equ., we introduce new co-rotating variables:

$$\tilde{\sigma}_{eg} = \sigma_{eg}\, e^{i\omega t} \qquad \tilde{\sigma}_{ge} = \sigma_{ge}\, e^{-i\omega t}$$

$$\tilde{\sigma}_{ee} = \sigma_{ee} \qquad \tilde{\sigma}_{gg} = \sigma_{gg}$$

(note that this is like going into the frame that rotates with the driving field)

$$\dot{\tilde{\sigma}}_{ee} = \frac{i\Omega}{2}\left(\tilde{\sigma}_{eg} - \tilde{\sigma}_{ge}\right) - \Gamma\tilde{\sigma}_{ee}$$

$$\dot{\tilde{\sigma}}_{gg} = \frac{-i\Omega}{2}\left(\tilde{\sigma}_{eg} - \tilde{\sigma}_{ge}\right) + \Gamma\tilde{\sigma}_{ee}$$

$$\dot{\tilde{\sigma}}_{ge} = -i\omega\, \underbrace{\sigma_{ge}\, e^{-i\omega t}}_{\tilde{\sigma}_{ge}} + \dot{\sigma}_{ge}\, e^{-i\omega t} =$$

$$-i\omega\tilde{\sigma}_{ge} + i\omega_0\, \sigma_{ge} e^{-i\omega t} - \frac{i\Omega}{2}\left(\tilde{\sigma}_{ee} - \tilde{\sigma}_{gg}\right) e^{i\omega t} \cdot e^{-i\omega t} - \frac{\Gamma}{2}\sigma_{ge} e^{-i\omega t}$$

$$\dot{\tilde{\sigma}}_{ge} = -i(\omega-\omega_0)\tilde{\sigma}_{ge} - \frac{i\Omega}{2}\left(\tilde{\sigma}_{ee} - \tilde{\sigma}_{gg}\right) - \frac{\Gamma}{2}\tilde{\sigma}_{ge}$$

similarly

$$\dot{\tilde{\sigma}}_{eg} = i(\omega-\omega_0)\tilde{\sigma}_{eg} + \frac{i\Omega}{2}\left(\tilde{\sigma}_{ee} - \tilde{\sigma}_{gg}\right) - \frac{\Gamma}{2}\tilde{\sigma}_{eg}$$

defining, as usual $\delta = (\omega - \omega_0)$ :

$$\dot{\tilde{\sigma}}_{ee} = \frac{i\Omega}{2}\left(\tilde{\sigma}_{eg} - \tilde{\sigma}_{ge}\right) - \Gamma \tilde{\sigma}_{ee}$$

$$\dot{\tilde{\sigma}}_{gg} = \frac{-i\Omega}{2}\left(\tilde{\sigma}_{eg} - \tilde{\sigma}_{ge}\right) + \Gamma \tilde{\sigma}_{ee}$$

$$\dot{\tilde{\sigma}}_{ge} = -i\delta\tilde{\sigma}_{ge} - i\frac{\Omega}{2}\left(\tilde{\sigma}_{ee} - \tilde{\sigma}_{gg}\right) - \frac{\Gamma}{2}\tilde{\sigma}_{ge}$$

$$\dot{\tilde{\sigma}}_{eg} = i\delta\tilde{\sigma}_{eg} + i\frac{\Omega}{2}\left(\tilde{\sigma}_{ee} - \tilde{\sigma}_{gg}\right) - \frac{\Gamma}{2}\sigma_{eg}$$

with normalization $(\sigma_{ee} + \sigma_{gg} = 1)$ and $\sigma_{eg} = \sigma_{ge}^*$, we have only 3 independent equations, so we will re-write these in a way conducive to interpretation as the vector model.

define:
$$u = \frac{1}{2}\left(\tilde{\sigma}_{ge} + \tilde{\sigma}_{eg}\right)$$

$$v = \frac{1}{2i}\left(\tilde{\sigma}_{ge} - \tilde{\sigma}_{eg}\right)$$

$$w = \frac{1}{2}\left(\sigma_{ee} - \sigma_{gg}\right)$$

differentiating these equations, and using the equations above, we get :

$$\dot{u} = \frac{-i}{2}\delta\left(\tilde{\sigma}_{ge} - \tilde{\sigma}_{eg}\right) + 0 \quad - \frac{\Gamma}{2}\cdot\frac{1}{2}\left(\tilde{\sigma}_{ge} + \tilde{\sigma}_{eg}\right)$$

$$u = \delta v - \Gamma/2\, u$$

$$\dot{v} = -\frac{\delta}{2}\left(\tilde{\sigma}_{ge} - \tilde{\sigma}_{eg}\right) - \frac{1}{2}\Omega\left(\tilde{\sigma}_{ee} - \tilde{\sigma}_{gg}\right) - \frac{\Gamma}{2}\frac{1}{2i}\left(\tilde{\sigma}_{ge} - \tilde{\sigma}_{eg}\right)$$

$$\dot{v} = -\delta u - \Omega w - \frac{\Gamma}{2} v$$

$$\dot{w} = \frac{i\Omega}{2}\left(\tilde{\sigma}_{eg} - \tilde{\sigma}_{ge}\right) - \Gamma\tilde{\sigma}_{ee} \qquad \text{using } \sigma_{ee} = w + \frac{1}{2}$$

$$\dot{w} \quad \Omega v - \Gamma w - \frac{\Gamma}{2}$$

summarizing:

$$\dot{u} = \delta v - \Gamma/2\, u$$

$$\dot{v} = -\delta u - \Omega w - \Gamma/2\, v$$

$$\dot{w} = \Omega v - \Gamma w - \Gamma/2 \quad = \quad \Omega v - \Gamma\left(w + 1/2\right)$$