So far, we have treated excitation of a \( 2 \)-level atom in the spirit of Fermi's Golden Rule. Now, we will go beyond that.

Using the eqs. derived earlier (p. 3 of lecture #3):

\[
i \hbar \dot{b}_g(t) = b_e(t) e^{-i\omega_0 t} \langle g | d, l | e \rangle
\]

\[
i \hbar \dot{b}_e(t) = b_g(t) e^{i\omega t} \langle e | d, l | g \rangle
\]

where \( |\Psi(t)\rangle = b_g(t) |g\rangle + b_e(t) e^{-i\omega t} |e\rangle \)

\[
\hat{H}_1(t) = \frac{\hbar}{2i} \left[ e^{i\omega t} - e^{-i\omega t} \right]
\]

As in the F.G.R. derivation we keep only the resonant terms:

\[
b_g = \frac{1}{2\hbar} e^{i(\omega - \omega_0) t} \hat{V}_{gg} b_e(t)
\]

\[
b_e = \frac{1}{2\hbar} e^{-i(\omega - \omega_0)} \hat{V}_{ge} b_g(t)
\]
Details of p. 1 derivation

\[ b_g(t) = \frac{1}{\varepsilon h} b_e(t) e^{-i\omega t} \left[ e^{i\omega t} - e^{-i\omega t} \right] \]

\[ b_e(t) = \frac{1}{\varepsilon h} b_g(t) e^{i\omega t} V_{eg} \left[ e^{i\omega t} - e^{-i\omega t} \right] \]

\[ b_g(t) = -\frac{1}{2\varepsilon h} V_{ge} \left[ e^{i(w-\omega)t} - e^{i(w+\omega)t} \right] \]

\[ b_e(t) = -\frac{1}{2\varepsilon h} V_{eg} \left[ e^{i(w+\omega)t} - e^{i(w-\omega)t} \right] \]

Throwing away the rapidly oscillating terms in \((w+\omega)\) we have.

\[ b_g(t) = -\frac{1}{2\varepsilon h} e^{i(w-\omega)t} V_{ge} \]

\[ b_e(t) = \frac{1}{2\varepsilon h} e^{-i(w-\omega)t} V_{eg} \]

As on p. 1.

Note that for the change in \(b_g(t)\) we kept the positive frequency term, while for the change in \(b_e(t)\) we kept the negative frequency term.
We want to solve these coupled equations, which we do by taking another time derivative of one of them:

\[
\frac{d}{dt} \left[ \frac{db_g}{dt} \right] = -\frac{1}{2\hbar} c^{i(\omega - \omega_0)t} \hat{V}_{eg} b_e(t)
\]

Define \( \omega - \omega_0 = \delta \)

\[
\frac{d^2 b_g}{dt^2} = -\frac{1}{2\hbar} \hat{V}_{eg} \left[ i\delta \delta \delta c^{i\delta \delta \delta \delta} b_e(t) + c^{i\delta \delta \delta \delta} \left( \frac{V_{ge}}{\hbar} \right) c^{i\delta \delta \delta \delta} b_g(t) \right]
\]

\[= \hat{b}_e(t)
\]

\[
b_g(t) = -\frac{1}{4\hbar^2} \hat{V}_{eg} V_{ge} b_g(t) - i\frac{\delta}{2\hbar} c^{i\delta \delta \delta \delta} b_e(t) \hat{V}_{eg}
\]

For the special case of \( \delta = 0 \), which is on-resonance driving of the transition:

\[
b_g(t) = -\frac{1}{4\hbar^2} |V_{ge}|^2 b_g(t) \quad \text{recall} \quad V_{ge} = V_{eg}^* \]

Define \( R_{\text{Rabi}} = \frac{|V_{ge}|}{\hbar} \)

then:

\[
b_g + \frac{\delta^2}{4\hbar} b_g = 0
\]
let us solve this with $b_g(0) = 1$, $b_e(0) = 0$

solution (by inspection)

$$b_g(t) = \cos\left(\frac{\Omega t}{2}\right)$$

$$P_g = |b_g(t)|^2 \quad P_e = |b_e(t)|^2 = 1 - P_g$$

$$P_g(t) = \cos^2\left(\frac{\Omega t}{2}\right) \quad P_e(t) = \sin^2\left(\frac{\Omega t}{2}\right)$$

the populations oscillate at the Rabi frequency

Rabi oscillations or Rabi flopping

Note that other conventions for defining the perturbation might be used e.g.

$$\Delta \hat{H} = 2\hat{V}\sin\omega t$$ instead of $\Delta \hat{H} = \Delta \hat{V} \sin\omega t$ to make $\Delta \hat{V}$ be the "essential" part of $\Delta \hat{H}$. But $\omega$ is the Rabi frequency.
If we let $\varepsilon \to 0$ we have

$$P_c(t) = \frac{|\text{veg}|^2}{4\varepsilon^2} \varepsilon^2 = \frac{\varepsilon^2}{4}$$

compare this with lecture #3 p. (5)

$$P_{yoe}(t) = \frac{|\text{veg}|^2}{4\varepsilon^2} \frac{\sin^2\left((\omega_0 - \omega)/2\right)}{\left[(\omega_0 - \omega)/2\right]^2}$$

Letting $(\omega_0 - \omega) = -\delta \to 0$ and using l'Hopital's rule

$$P_{yoe}(t) = \frac{|\text{veg}|^2}{4\varepsilon^2} \varepsilon^2$$

in agreement with what we have just derived.

It can be shown (and it is likely that we will later show) that for $\delta \neq 0$

$$P_c(t) = \frac{\delta^2}{2(\delta^2 + \delta^2)} \left[1 - \cos\left((\omega^2 + \delta^2)^{1/2} \right) \right]$$

or

$$P_c(t) = \frac{\delta^2}{\sigma^2 + \delta^2} \sin^2\left((\omega^2 + \delta^2)^{1/2} \right)$$

which agrees with $P_c(t)$ for $\delta = 0$ on p. 3
We define

$$\Omega_{\text{eff}} = \sqrt{p^2 + s^2}$$

as the "effective" or "generalized" Rabi frequency.

Note that for $s=0$, the population oscillates between 0 and 1.

But for $s \neq 0$, it oscillates between 0 and $\frac{s^2}{p^2 + s^2}$.

(Remember that, so far, there is no decay of 1e, and no spread of $\omega$, i.e., of $s$.)
There is a simple interpretation of this resonant $\pm \omega$ off resonant behavior, and we will derive it, rigorously (more or less).

But first, let us take an overall view of it, non-rigorously.

Recalling that a spin $\frac{1}{2}$ in a magnetic field is a 2-level system, and all 2-level systems are mathematically equivalent, let us think of this as a spin.

Also, recalling that the expectation value of a Q.M. observable follows the eq. of motion of the corresponding classical observable (Ehrenfest theorem), let us consider a classical spin.

(we will prove this for our Q.M. spin)
It is a classical spin magnetic moment. Classically, it can point anywhere. Q.M. if in an energy eigenstate, it is up or down. If not in an eigenstate, it is not "stationary." Classically, if not up or down, it precesses:

\[ \dot{\mathbf{S}} = \gamma \mathbf{S} \times \mathbf{B} \]

\[ |\mathbf{L}| \text{ is the magnetic moment } \]

\[ \mathbf{L} \text{ is the torque} \]

As in the case of a gyroscope the precession frequency

\[ \omega_0 = \gamma |\mathbf{B}| \]

is independent of the angle. Precession occurs at a constant constant angle.
the quantum analog is for a general state:

\[ |\Psi(\tau)\rangle = b^* |g\rangle + b e^{-i\omega_0 \tau} |e\rangle \quad b^2 + b^* g^2 = 1 \]

this state evolves at frequency \( \omega_0 \), and

 corresponds to rotation of the \( x-y \) projection \( \langle \hat{\sigma}_y \rangle \) at frequency \( \omega_0 \), with \( |b_{x,y}| \approx |b^* g e^{i\omega_0 \tau}| \)

This all describes the free evolution with \( \mathcal{H}_1 = 0 \)

(\( B_0 \) is just the B-field \( \vec{B}_0 = -B_0 \hat{z} \)).

To couple \( |g\rangle \) and \( |e\rangle \) to \( |v\rangle \) and \( |f\rangle \) we need another field to give an \( \mathcal{H}_1(t) \). This field is \( \vec{B}_1(t) \), in the \( x-y \) plane, rotating at frequency \( \omega_1 \).
It is easier to treat this problem in a rotating frame, rotating with \( \vec{B} \) at \( \omega \), around \( \vec{Z} \) \\
\text{(this is a bit like the transformation we made from} \\
\quad c_\omega(t) \rightarrow b_\omega(t) e^{-i\omega t} \text{, but not exactly)} \\
\text{Transformation to the rotating (non-inertial) frame introduces a "fictitious" (frame) torque of} \\
\quad \vec{T} = \vec{\omega} \times \vec{\omega}/8 \\
\text{[To see this, take } \omega = \omega_0 \text{ with } b_\omega = 0 \text{, then there is no torque + no precession because there is effectively an additional field } \vec{\omega}/8 \text{ that cancels } B_0 \text{]} \\
\text{in general, we have a new effective field} \\
\quad B_0' = \frac{\omega}{8} - B_0 \quad \text{(recall that } \vec{B}_0 \text{ is along } -\vec{Z} \text{ in our example)} \\
\quad B_0' = (\omega - \omega_0)/8 \\
\text{(this is the Lorenz theorem)} \\
\quad \gamma B_0' = \delta
For $\omega < \omega_0$ we would have (taking $\vec{B}_1$ along $x$ in the rotating frame):

(rotating)

\[
\vec{B}_{\text{eff}} = (\vec{B}_1 + \vec{B}_0')/2
\]

\[
\delta B_{\text{eff}} = (\Omega^2 + \delta^2)^{1/2}
\]

where $\delta = \delta B_1$

At resonance ($\omega = \omega_0$) $B_0' = 0$, $\vec{B}_{\text{eff}} = \vec{B}_1$

It precesses about $\vec{B}_1$
at frequency $\omega = \delta B_1$
(the Rabi frequency)
On resonance the moment \( \vec{m} \) starts at \(-\hbar\) (i.e., all in ground state) proceeds all the way to \(+\hbar\) (i.e., all in excited state) and back at the Rabi frequency.

Off resonance:

\( \vec{m} \) precesses rapidly, at \( \delta_{\text{eff}} = \sqrt{\delta^2 + \Delta^2} \), but with little amplitude, never getting much into the excited state.
the angle of $\mathbf{B}_{\text{eff}}$ with $-\mathbf{z}$ is:

\[ \cos \theta = \frac{B_0^\prime}{\sqrt{B_0^2 + B_0^\prime^2}} \]

\[ \sin \theta = \frac{\delta}{\sqrt{\omega^2 + \delta^2}} \]

The excited-state population is given by

\[ \frac{1}{2} (1 - \cos 2\theta) = \frac{1}{2} \left( 1 - \left[ 1 - 2\sin^2 \theta \right] \right) = \sin^2 \theta \]

so

\[ P_e(t) = \frac{\omega^2}{\sqrt{\omega^2 + \delta^2}^2} \left[ 1 - \cos \left( \sqrt{\omega^2 + \delta^2} \right)^2 t \right] \]

as we wrote on p. 41

Now, let us do this quantum mechanically.

It will be convenient to use the density matrix formulation.