Lecture #6 20 Sept. 2005

So far, we have treated excitation of a 2-level atom in the spirit of Fermi's Golden Rule. Now, we will go beyond that.

using the eqs. derived earlier (p. 3 of lecture #3):

$$i\hbar \dot{b}_g(t) = b_e(t)\, e^{-i\omega_0 t} \langle g|\mathcal{H}_1|e\rangle$$

$$i\hbar \dot{b}_e(t) = b_g(t)\, e^{i\omega t} \langle e|\mathcal{H}_1|g\rangle$$

where $|\Psi(t)\rangle = b_g(t)|g\rangle + b_e(t)e^{-i\omega_0 t}|e\rangle$

and $\mathcal{H}_1(t) = \frac{\hat{V}}{2i}\left[e^{i\omega t} - e^{-i\omega t}\right]$

as in the F.G.R. derivation we keep only the resonant terms:

$$\dot{b}_g = -\frac{1}{2\hbar} e^{i(\omega-\omega_0)t}\, \hat{V}_{eg}\, b_e(t)$$

$$\dot{b}_e = \frac{1}{2\hbar} e^{-i(\omega-\omega_0)}\, \hat{V}_{ge}\, b_g(t)$$

details of p. 1 derivation

$$\dot{b}_g(t) = \frac{1}{i\hbar} b_e(t)\, e^{-i\omega_0 t}\, V_{ge} \cdot \frac{1}{2i} \left[e^{i\omega t} - e^{-i\omega t} \right]$$

$$b_e(t) = \frac{1}{i\hbar} b_g(t)\, e^{i\omega_0 t}\, V_{eg} \cdot \frac{1}{2i} \left[e^{i\omega t} - e^{-i\omega t} \right]$$

$$\dot{b}_g(t) = -\frac{1}{2\hbar} V_{ge} \left[e^{i(\omega-\omega_0)t} - e^{-i(\omega+\omega_0)t} \right]$$

$$\dot{b}_e(t) = -\frac{1}{2\hbar} V_{eg} \left[e^{i(\omega+\omega_0)t} - e^{-i(\omega-\omega_0)t} \right]$$

throwing away the rapidly oscillating terms in $(\omega+\omega_0)$ we have.

$$\dot{b}_g(t) = -\frac{1}{2\hbar} e^{i(\omega-\omega_0)t} V_{ge}$$

$$\dot{b}_e(t) = \frac{1}{2\hbar} e^{-i(\omega-\omega_0)t} V_{eg}$$

as on p. 1.

Note that for the change in $b_g(t)$ we kept the <u>positive</u> frequency term, while for the change in $b_e(t)$ we kept the <u>negative</u> frequency term.

We want to solve these coupled equations, which we do by taking another time derivative of one of them:

$$\frac{d}{dt}\left[\frac{d\,b_g}{dt} = -\frac{1}{2\hbar}\, e^{i(\omega-\omega_0)t}\, \hat{V}_{eg}\, b_e(t)\right]$$

define $\omega - \omega_0 \equiv \delta$

$$\frac{d^2 b_g}{dt^2} = -\frac{1}{2\hbar} V_{eg}\left[i\delta\, e^{i\delta t}\, b_e(t) + \underbrace{e^{i\delta t}\left(\frac{V_{ge}}{2\hbar}\right) e^{-i\delta t}\, b_g(t)}_{=\,\dot{b}_e(t)}\right]$$

$$\ddot{b}_g(t) = -\frac{1}{4\hbar^2} V_{eg} V_{ge}\, b_g(t) \quad \frac{-i\delta}{2\hbar}\, e^{i\delta t}\, b_e(t)\, V_{eg}$$

for the special case of $\delta = 0$, which is on-resonance driving of the transition:

$$\ddot{b}_g(t) = \frac{-1}{4\hbar^2} |V_{ge}|^2\, b_g(t) \qquad \text{recall } V_{ge} = V_{eg}^*$$

define $\Omega_{Rabi} = \dfrac{|V_{ge}|}{\hbar}$

then:

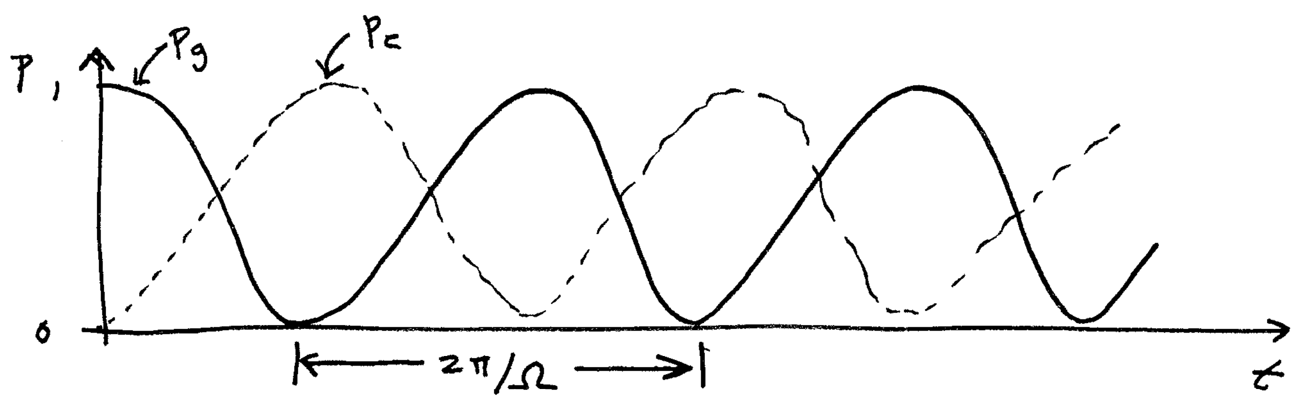
$$\ddot{b}_g + \frac{\Omega^2}{4} b_g = 0$$

let us solve this with $b_g(0) = 1$, $\dot{b}_g(0) = 0$

solution (by inspection)

$$b_g(t) = \cos\left(\frac{\Omega t}{2}\right)$$

$$P_g = |b_g(t)|^2 \qquad P_e = |b_e(t)|^2 = 1 - P_g$$



$$P_g(t) = \cos^2\left(\frac{\Omega t}{2}\right) \qquad P_e(t) = \sin^2\left(\frac{\Omega t}{2}\right)$$

the populations oscillate at the Rabi frequency

Rabi oscillations or
Rabi flopping

Note that other conventions for defining the perturbation might be used e.g.

$\mathcal{H}_1 = 2\hat{V}\sin\omega t$ instead of $\mathcal{H}_1 = \hat{V}\sin\omega t$ to make V be the "effective" part of $\mathcal{H}$. But Ω is the flopping freq.

If we let $t \to 0$ we have

$$P_e(t) = \frac{|V_{eg}|^2}{4\hbar^2} t^2 = \frac{\Omega^2 t^2}{4}$$

compare this with lecture #3 p. 5

$$P_{g\to e}(t) = \frac{|V_{eg}|^2}{4\hbar^2} \frac{\sin^2(\omega_0-\omega)t/2}{[(\omega_0-\omega)/2]^2}$$

Letting $(\omega_0-\omega) = -\delta \to 0$ and using l'Hôpital's rule

$$P_{g\to e}(t) = \frac{|V_{eg}|^2 t^2}{4\hbar^2}$$ in agreement with what we have just derived.

It can be shown (and it is likely that we will later show) that for $\delta \neq 0$

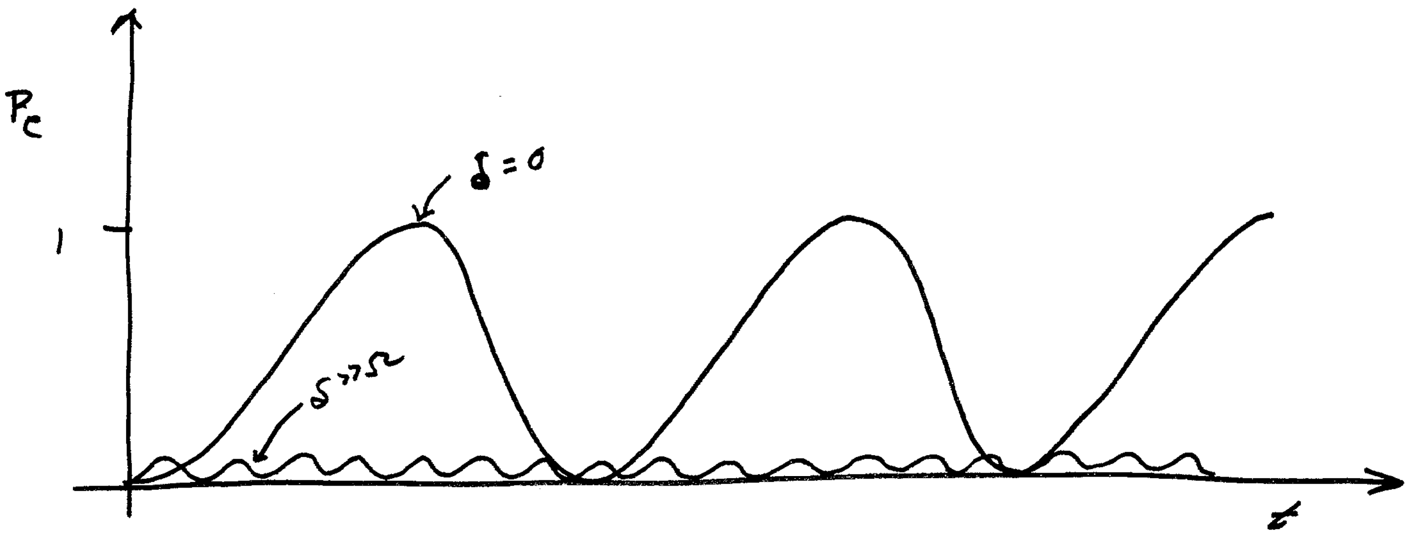
$$P_e(t) = \frac{\Omega^2}{2(\Omega^2+\delta^2)}\left[1-\cos(\Omega^2+\delta^2)^{1/2} t\right]$$ or

$$P_e(t) = \frac{\Omega^2}{\Omega^2+\delta^2} \sin^2 \frac{(\Omega^2+\delta^2)^{1/2}}{2} t$$, which agrees with $P_e(t)$ for $\delta = 0$ on p. 3

We define

$$\Omega_{eff} = \sqrt{\Omega^2 + \delta^2}$$

as the "effective" or "generalized" Rabi frequency



Note that for $\delta = 0$, the population oscillates between 0 and 1

but for $\delta \neq 0$, it oscillates between 0 and $\frac{\Omega^2}{\Omega^2 + \delta^2}$

(Remember that, so far, there is no decay of $|e\rangle$, and no spread of ω, i.e., of δ)

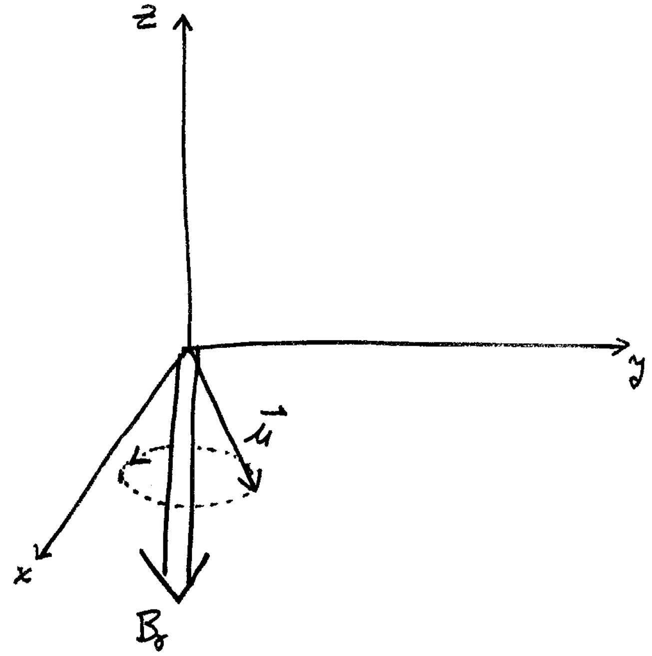
There is a simple interpretation of this resonant + off resonant behavior, and we will derive it, rigorously (more or less).

But first, let us take an overall view of of it, non-rigorously

Recalling that a spin 1/2 in a magnetic field is a 2-level system, and all 2-level systems are mathematically equivalent, let us think of this as a spin.

Also, recalling that the expectation value of a Q.M. observable follows the eq. of motion of the corresponding classical observable (Ehrenfest thm.), let us consider a classical spin.

(we will prove this for our Q.M. spin)



$\vec{\mu}$ is a classical spin magnetic moment.
classically it can point anywhere
Q.M., if in an energy eigenstate, it is up or down
if ~~is~~ not in an eigenstate, it is not "stationary"

classically, if not up or down, it precesses:

$$\dot{\vec{\mu}} = \gamma \underbrace{\vec{\mu} \times \vec{B}}_{\sim \text{ torque}}$$

$|\mu|$ is the magnetic moment

As in the case of a gyroscope the precession frequency

$\omega_0 = \gamma |B|$ is independent of the angle

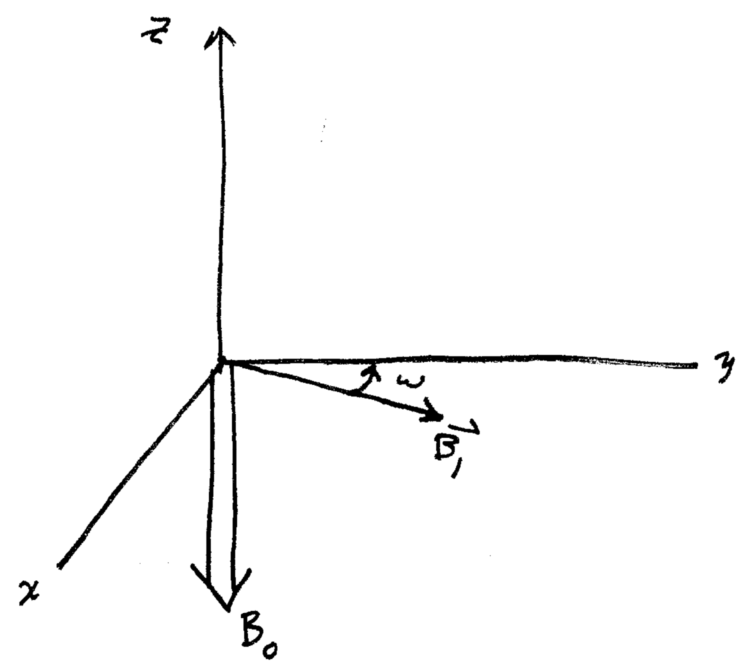
precession occurs at a ~~constant~~ constant angle the quantum analog is for a general state.

$$|\psi(t)\rangle = b_g|g\rangle + b_e e^{-i\omega_0 t}|e\rangle \qquad b_e^2 + b_g^2 = 1$$

this state evolves at frequency ω_0, and corresponds to rotation of the x-y projection of $\langle\vec{\mu}\rangle$ at frequency ω_0, with $|\mu_{xy}| \sim |b_g b_e^*|$

this all describes the free evolution with $\mathcal{H}_1 = 0$ ($\mathcal{H}_0$ is just the B-field $\vec{B}_0 = -B_0\hat{z}$.)

To couple $|g\rangle$ and $|e\rangle$ or $|\downarrow\rangle$ and $|\uparrow\rangle$ we need another field to give an $\mathcal{H}_1(t)$. this field is $\vec{B}_1(t)$, in the x-y plane, rotating at frequency ω:



It is easier to treat this problem in a rotating frame, rotating with $\vec{B}_1$ at ω, around $\hat{z}$

(this is a bit like the transformation we made from $c_e(t) \Rightarrow b_e(t)\, e^{-i\omega_0 t}$, but not exactly)

Transformation to the rotating (non-inertial) frame ~~the~~ introduces a "fictitious" ("frame") torque of

$$T = \vec{\mu} \times \vec{\omega}/\gamma$$

[To see this, take $\omega = \omega_0$ with $B_1 = 0$. then there is no torque + no precession because there is effectively an additional field ω/γ that cancels B_0]

in general, we have a new effective field

$$B_0' = \omega/\gamma - B_0$$
$$B_0' = (\omega - \omega_0)/\gamma$$

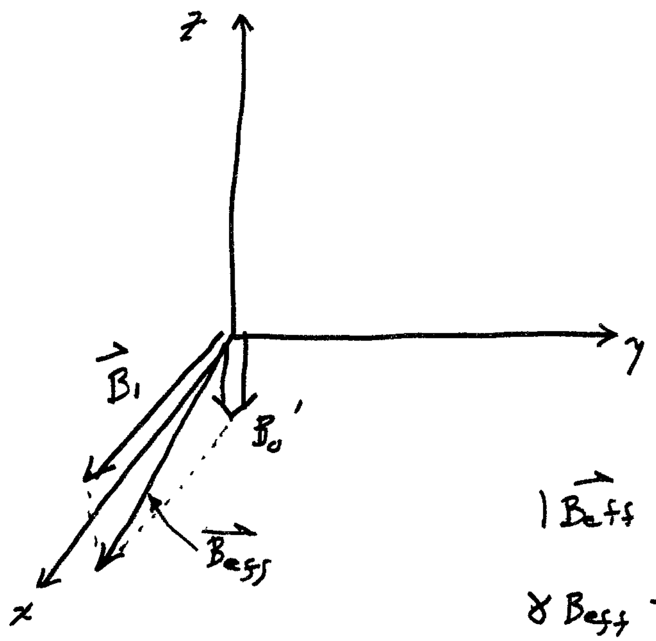
(recall that $\vec{B}_0$ is along $-\hat{z}$ in our example.)

(this is the Larmor theorem)

$$\gamma B_0' \equiv \delta$$

For $\omega < \omega_0$ we would have (taking $\vec{B}_1$ along x in the rotating frame.)

(rotating)

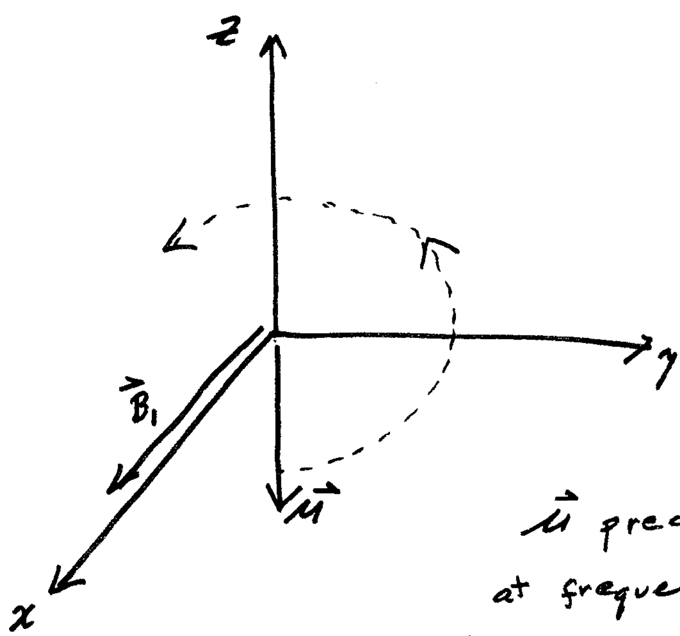


$$|\vec{B}_{eff}| = (B_1^2 + B_0'^2)^{1/2}$$

$$\gamma B_{eff} = (\Omega^2 + \delta^2)^{1/2}$$

where $\Omega = \gamma B_1$

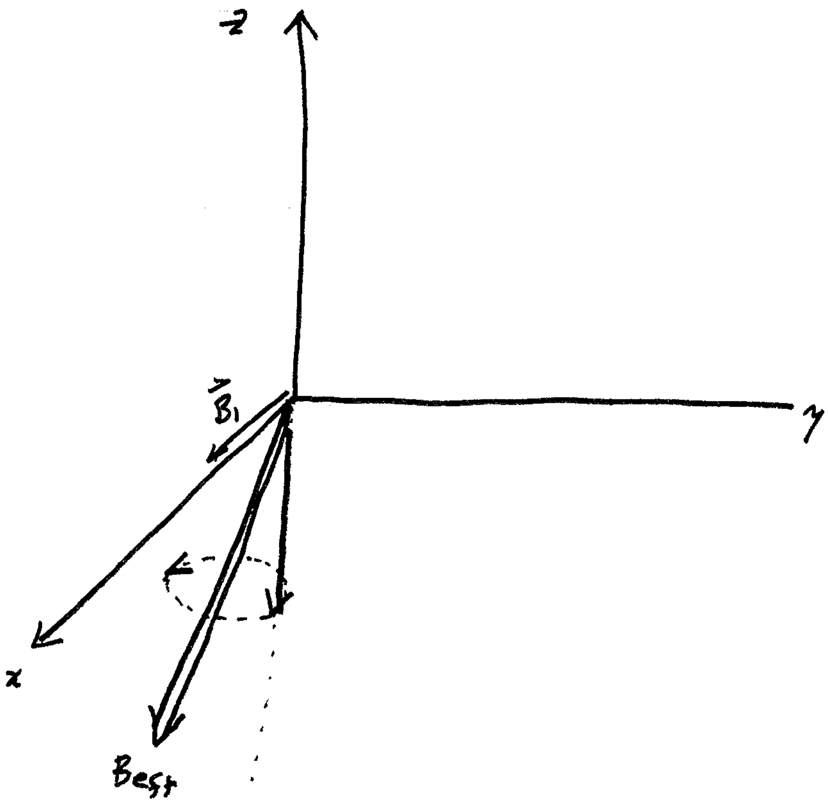
on resonance $(\omega = \omega_0)$ $B_0' = 0$, $\vec{B}_{eff} = \vec{B}_1$



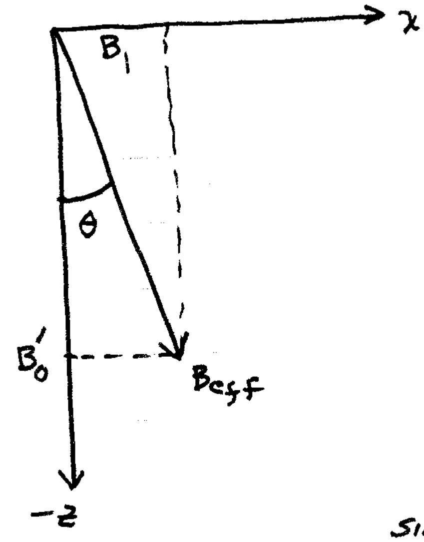
$\vec{\mu}$ precesses about $\vec{B}_1$ at frequency $\Omega = \gamma B_1$ (the Rabi frequency)

On resonance the moment $\vec{\mu}$, starting at $-\vec{z}$ (i.e., all in gnd state) precesses all the way to $+\vec{z}$ (i.e., all in excited state) and back, at the Rabi frequency.

Off resonance:



$\vec{\mu}$ precesses rapidly, at $\Omega_{eff} = \sqrt{\Omega^2 + \delta^2}$, but with little amplitude, never getting much into the excited state the angle of $\vec{B}_{eff}$ with $-\vec{z}$ is :



$$\cos\theta = \frac{B_0'}{\sqrt{B_1^2 + B_0'^2}}$$

$$= \frac{\delta}{\sqrt{\Omega^2+\delta^2}}$$

$$\sin\theta = \frac{\Omega}{\sqrt{\Omega^2+\delta^2}}$$

the maximum excited-state population is given by

$$\frac{1}{2}(1-\cos 2\theta) = \frac{1}{2}\left(1-\left[1-2\sin^2\theta\right]\right) = \sin^2\theta$$

so $$P_e(t) = \frac{\Omega^2}{\Omega^2+\delta^2}\ \frac{\left[1-\cos(\Omega^2+\delta^2)^{1/2}t\right]}{2}$$

as we wrote on p. 4

Now, let us do this quantum mechanically

It will be convenient to use the density matrix formulation.