Now look at emission as absorption from a microscopic, quantum point of view.

Consider two isolated atomic levels $|e\rangle, |g\rangle$, assume they interact via the electric dipole term: $\vec{d} \cdot \vec{E}$. \[ E_{e} - E_{g} = \hbar \omega. \]

**Emission**: assume the initial state is the excited atomic state, with no photons excited, and the final state is the atomic ground state, plus one photon in some direction $\hat{\lambda}$.

$|e, 0, 0, 0 \rightarrow \rightarrow |g, 0, 0, 1_{\lambda}, 0, 0 \rightarrow \rightarrow$

Use Fermi's golden rule: In principle, there is an angular dependence to the matrix elements $\vec{d} \cdot \vec{E}$, so one should calculate the rate into a solid angle $d\Omega$ and integrate.

Treating the sum over $\hat{\lambda}$ as an integral:

\[ R_{\text{reg}} = \int \frac{d\Omega_{\text{reg}}}{d\Omega_{\text{reg}}} \frac{d\omega_{2}}{d\omega_{2}} d\omega_{1} \]

where

\[ \frac{d\Omega_{\text{reg}}}{d\omega_{2}} = \frac{2\pi}{\hbar} |< g, 1\lambda | \hat{d} \cdot \hat{E}_{\lambda} | e, 0 > |^{2} \rho(E) \delta(E - \hbar \omega) \]

$\rho(\hbar \omega)$ is the density of photon modes $\frac{V}{(2\pi)^{3}} \frac{\hbar^{2} c^{3}}{\hbar^{2} c^{3}}$.

$\rho(\hbar \omega) = \frac{V}{(2\pi)^{3}} \frac{\hbar^{2} c^{3}}{\hbar^{2} c^{3}}$ \[
\int d\omega \text{ the } E \text{ integral, } |\hat{\lambda}| = \frac{\hbar \omega}{c}
\]
\[ R_{eq} = \sum_{k} \frac{2\pi}{k} \left| \langle g | c | e \rangle \langle E_k \rangle \langle \hat{a} \rangle \right|^2 \left( \frac{V}{(2\pi)^3 \hbar c^3} \right) \frac{\omega_0^2}{2 \epsilon_0 \nu} \frac{d\Omega}{d\Omega} \]  

replace this with its average over orientations: \( \langle \hat{a} \rangle = 1 \)  

\[ \frac{1}{3} \left| \langle g | \hat{d} | e \rangle \right|^2 \]  

\[ e^2 Me^2, \hat{M}_i = \frac{1}{Z} \hat{R}_i \]  

\[ \frac{2\pi}{k} \frac{e^2 Me^2}{3} \frac{\hbar \omega_0}{2 \epsilon_0 \nu} \frac{V}{(2\pi)^3} \frac{\omega_0^2}{\hbar c^3} 2 \times 4\pi \]

\[ = \frac{e^2 Me^2 \omega_0^3}{3 \pi \hbar \epsilon_0 c^3} = A_{eq} \]

note how easy this was with the quantized field!  

Also note that \( A_{eq} = \frac{1}{\tau} \propto \omega_0^3 \)  

excited state lifetime \( \tau \)  

this is because the phase space \( \propto \omega_0^2 \)  

and the coupling strength \( \propto \omega_0 \)  

(at least for a fixed size atom)
What if there had been a large number of photons in one mode to start with?

$\rightarrow$ stimulated emission

e.g. initial state $| \rangle = |e, 0, \ldots n_{kx}, 0 \ldots \rangle$

then for that mode, $\hat{a}^\dagger \vec{E}_{kx} = \hat{a}^\dagger \vec{E}_{kx} \left( \hat{a}_{kx} - \hat{a}_{kx}^+ \right) E_{0k}$

will couple

$\langle g, 0, \ldots \hat{n}_{kx} + 1, 0 \ldots | \hat{a}_{kx}^\dagger | e, 0, \ldots \hat{n}_{kx}, 0 \ldots \rangle \rightarrow$ energy

and

$\langle g, 0, \ldots \hat{n}_{kx} + 1, 0 \ldots | \hat{a}_{kx} | e, 0, \ldots \hat{n}_{kx}, 0 \ldots \rangle \rightarrow$ doesn't conserve energy.

the full $\hat{a}_{kx}$ term gives

$$\left| E_{0k} \right|^2 \left| g \hat{d} \cdot \vec{E}_{kx} \right|^2 \left( \langle 0, \ldots \hat{n}_{kx} + 1, 0 \ldots | \hat{a}_{kx}^\dagger | e, 0, \ldots \hat{n}_{kx}, 0 \ldots \rangle \right)^2$$

We have to be careful using Fermi's Golden Rule, because there are two states coupled, not a continuum of states. Fermi's Golden rule only works when coupling to a continuum of states.

We can discuss the problem in several ways:
1) Solve the two-state problem, which will be described by Rabi oscillations, not Fermi's Golden Rule. This assumes we ignore the coupling of \( 1e > \) to other modes.

2) Assume the input radiation is broad-band in energy, filling a continuum in energy around the resonant frequency of the atom. (Similar to Einstein's approach)

3) Ask "How did the atom get to \( 1e > \) in the first place?" and consider the full scattering process: \( 1g > \xrightarrow{\text{photon}} 1e > \xrightarrow{\text{photon}} 1g > \)

(Note for next weeks lecture: the Optical Bloch equations provide a framework for discussing spontaneous processes (Fermi's Golden Rule) and Rabi oscillations between 2 levels simultaneously)

For now take approach #2

Assume the light propagates along a direction \( \hat{k} \), but has a spread in frequency (energy/\( \hbar \)) with a constant # of photons in each mode \( n_0 \):

\[
\text{# of photons between } \omega, \omega + \Delta \omega, \text{ in solid angle } d\Omega = n_0 \int \rho_d(\omega) \delta(2k_x - 2k) \Delta \omega
\]

\[
= n_0 \int_\omega \rho_d(E) \delta(2k_x - 2k) dE.
\]

Here, \( \rho_d(E) \) is the 1d mode density, \( n_0 \) is the average number of photons in the mode (along \( \hat{k} \)).
Now apply Fermi's Golden Rule

\[ \mathcal{R}_e = \int \frac{2\pi}{\hbar} |\langle \text{final} | \mathbf{d} \cdot \mathbf{E} | e, o, \ldots n_{k_f}, 0 \rightarrow \mathbf{k} \rangle|^2 \rho(E) \delta(E) \, dE \, d\Omega_k \]

there are two different types of final states:

1) \( \mathbf{k}_f \) along \( \mathbf{k}' \), for which the matrix element involves the state \( |g, o, \ldots n_{k_f} - 1, o \ldots \rangle \):

\[ |\langle g|d|e\rangle \cdot \mathbf{\hat{E}}_{k_{f'}}|^2 \quad E_{0k}^2 \quad (N_0 + 1) \rho_{1d}(\omega_0) \]

2) \( \mathbf{k}_f \) not along \( \mathbf{k}' \), for which the matrix element involves the state \( |g, o, \ldots, l_{k_f}, n_{k_f}, 0 \ldots \rangle \):

\[ |\langle g|d|e\rangle \cdot \mathbf{\hat{E}}_{k_{f'}}|^2 \quad E_{0k}^2 \quad 1 \rho_{3d}(\omega_0) \]

When we do the integral over \( dE \cdot d\Omega_k \), include the \(+1\) from \((N_0 + 1)\) with the other single photon spontaneous events.

\[ \mathcal{R}_e = \frac{2\pi}{\hbar} |\langle g|d|e\rangle \cdot \mathbf{\hat{E}}_{k_{f'}}|^2 E_{0k}^2 N_0 \rho(\hbar \omega_0) \]

\[ + \frac{\varepsilon^2 M_e^2}{3\pi \hbar^2 \varepsilon_0 C^3} \]

\[ = \text{this is stimulated emission} \]

\[ \frac{\varepsilon^2 M_e^2}{3\pi \hbar^2 \varepsilon_0 C^3} \]

\[ = \text{this is just the spontaneous emission} \]

\[ \text{Ag from before.} \]

\[ |\langle g|d|e\rangle \cdot \mathbf{\hat{E}}_{k_{f'}}|^2 = \frac{1}{3} \varepsilon^2 M_e^2, \quad M_e = \sum \hat{\mathbf{r}}_i \]

\[ = \text{assume that the ensemble of scattering atoms are pointed in random directions.} \]

\[ \sum \hat{\mathbf{r}}_i \]
\[ R_{eq} = \frac{2\pi}{h} \frac{e^2 M_g^2}{3} \frac{\hbar \omega_0}{260 V} n_o \rho_{id} + A_{eq} \]

\[ = \frac{\pi e^2 M_g^2}{60 \hbar^2} \frac{\hbar \omega_0 n_o \rho_{id}}{V} + A_{eq} \]

but \(\hbar \omega = \text{Energy per photon} \)
\(n_o \rho_{id} = \# \text{ of photons per } \delta w\)
\(V = \text{volume}\)

\[ R_{eq} = \frac{\pi e^2 M_g^2}{60 \hbar^2} \frac{\hbar \omega_0 n_o \rho_{id}}{V} + A_{eq} \]

\[ \frac{\hbar \omega_0 n_o \rho_{id}}{V} = W(w) \]
the energy density per \(\delta w\)

\[ B_{eq} \quad \text{and} \quad A_{eq} = \frac{\hbar \omega_0^3}{\pi^2 C^3} \]

As \(\omega_0\) increase, spontaneous emission becomes stronger relative to stimulated emission for fixed \(W(w)\)

Since \(A + B\) both depend on \(e^2 M_g^2\), the ratio is independent \(\to\) a fundamental result.
\( \frac{\hbar \omega_0^3}{\pi^2 c^3} \) has units of energy density per \( dw \)

defines a saturation energy density per \( dw \):

\[ W_s = \frac{\hbar \omega_0^3}{\pi^2 c^3} \]

There will be a corresponding saturation intensity per \( dw \):

\[ \rho_s = c W_s \]

Imagine shining (broad band) light on the atom from one direction.

What are the steady state energy level populations? What are the average scattering rates? (as a function of incoming energy density)

Rate equations:

\[ N_q = A N_e + B W(\omega)(N_e - N_q) \]

\[ N_e = -N_g \]

In steady state, \( \dot{N}_e = \dot{N}_g = 0 \) \( \Rightarrow \) \( N_{\text{tot}} = N_g + N_e \)

\[ N_g = \frac{A + B W(\omega)}{A + 2BW(\omega)} = \frac{W_s + W(\omega)}{W_s + 2W(\omega)} N_{\text{tot}} \]

\[ N_e = \frac{BW(\omega)}{A + 2BW(\omega)} = \frac{W(\omega)}{W_s + 2W(\omega)} N_{\text{tot}} \]
What about the total scattering rates?

Absorption: \[ N_g B W(\omega) = \frac{N \bar{A}}{\text{tot}} \frac{(W_s + W(\omega))(W(\omega))}{W_s + 2W(\omega)} \]

Stimulated emission: \[ N_e B W(\omega) = N_{\text{tot}} A \left( \frac{W(\omega)}{W_s + 2W(\omega)} \right) \]

Spontaneous emission: \[ N_e A = N_{\text{tot}} A \left( \frac{W(\omega)}{W_s + 2W(\omega)} \right) \]

\[ A = \frac{1}{2} \]

\[ \frac{\text{scattering rate} \times 2}{N_{\text{tot}}} \]

Absorption
Stimulated emission
Spontaneous scattering
See Bill's lecture notes from lecture #3 for a discussion of the cross section.

I did not get to this in class, but there is a useful classical analogy to scatter by:

Imagine an electron on a spring subject to an oscillatory $\mathbf{E}$-field:

\[
F = e\mathbf{E} \cos(\omega t)
\]

(Gaussian units, sorry)

Take $\hat{x}$ to be along the polarization direction.

For $\mathbf{E}$, $\mathbf{F} \cdot \mathbf{E} = x \mathbf{E}$

\[
\ddot{x} - \gamma \dot{x} + \omega_0^2 x = \frac{e}{m} E \cos(\omega t).
\]

(this has classical solutions (in steady state))

\[
x(t) = -\frac{eE}{m} \cos(\omega t) \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \]

\[
-\frac{eE}{m} \sin(\omega t) \frac{2\gamma \omega}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}
\]

\[
x(t) = \frac{eE}{m} \sin(\omega t) \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \]

\[
-\frac{eE}{m} \cos(\omega t) \frac{2\gamma \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}
\]
Note: the power dissipated is \( \int_0^T x(t)F(t) \, dt = xF \)
only the in-phase parts will contribute to the time-average.

the time average electrostatic energy is \( \int_0^T xF \, dt = xF \)
again, only the in phase part will contribute.

\[ x \propto \frac{1}{(W-W_0)^2} \quad \text{(for } \frac{x}{W_0} \ll 1) \]

\( \rightarrow \) in phase \( \propto \) power dissipated.

\( \rightarrow \) out of phase, no dissipation.

\( \rightarrow \) out of phase doesn't contribute to time-averaged energy.

\( \alpha \propto \frac{1-W_0}{W_0} \)

doesn't go to zero

\( \alpha \propto \text{total time averaged energy in electron/spring.} \)

\( \rightarrow \) Many properties of atoms will look similar.
Look on resonance:

$$\dot{XF} = \frac{e^2E^2}{m} \frac{1}{\gamma^4}$$

if $\frac{e^2E^2}{m}$ and $\gamma^4$ are independent,

we can get different widths and heights of the dissipation ("scattering") resonance:

$$\gamma = \frac{e^2ω_0^2}{3mc^3} \frac{1}{4πε_0}$$

(see Jackson for dipole radiation)

$$\dot{XF} = \frac{e^2E^2}{4m} \frac{3mc^3e^4i}{ω_0^2} = \frac{3}{4} \frac{e^2E^2c}{ω_0^2} \frac{c^24π}{1/k^2} \propto I \lambda^2$$

But for an electron oscillating, we know it classically radiates,

changing $\gamma$ also changes the coupling $\frac{eE}{m}$, so on resonance dissipation

is the same. $\Rightarrow$ more like atoms.