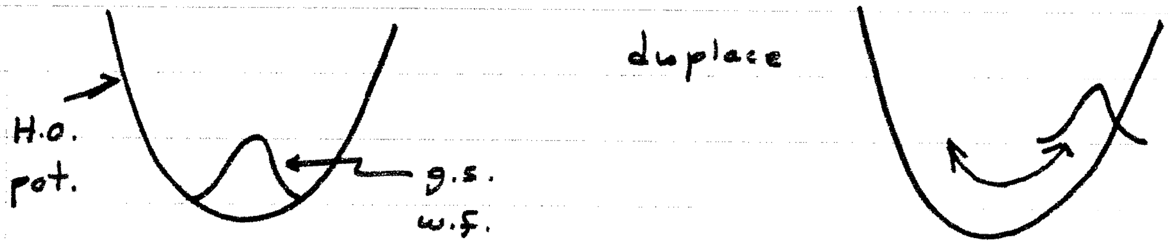
Physics 721 Lecture #3 8 sept 2005

Coherent states (from last lecture)



a displaced ground-state w.f. oscillates as an undistorted wavepacket, much like a classical particle. this coherent stat is

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n e^{-\frac{|\alpha|^2}{2}} |n\rangle}{\sqrt{n!}}$$

α is a complex number whose amplitude + phase are related to the amplitude + phase of oscillation

$$P_n \equiv |\langle n|\alpha\rangle|^2 = \left(|\alpha|^2\right)^n \frac{e^{-|\alpha|^2}}{n!},$$

which is identical to the Poisson distribution

$$\mathcal{P}_{\bar{n}}(n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!} \quad \text{with } |\alpha|^2 = \bar{n}$$

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topics: review of Fermi's "Golden Rule"
Einstein A+B coefficients
on-resonant cross section

take $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ $\mathcal{H}_0, \mathcal{H}_1$ are hermitian

assume $\mathcal{H}_0 |i\rangle = E_i |i\rangle$, the eigenvalue problem, has 2 solutions:

$$\mathcal{H}_0 |g\rangle = E_g |g\rangle$$

E_i s are <u>real</u>

$$\mathcal{H}_0 |e\rangle = E_e |e\rangle$$

this is a "two-level" (really, two-state) system

let $E_g = 0$, $E_e \equiv \hbar\omega_0$

any state of this system (atom) can be written:

$$\psi = c_g |g\rangle + c_e |e\rangle$$

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now, add the perturbation:

$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1(t)$ where $\mathcal{H}_1(t) = 0$ for $t < 0$

the system obeys:

$$\mathcal{H}|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle$$

we want to calculate, for $|\psi\rangle_{initial} = |g\rangle$,

$$P_{g\to e} = |\langle e|\psi(t)\rangle|^2 .$$

With $|\psi(t)\rangle = c_g(t)|g\rangle + c_e(t)|e\rangle$

$$(\mathcal{H}_0 + \mathcal{H}_1(t))|\psi\rangle = i\hbar \frac{d}{dt}\left[\; c_g(t)|g\rangle + c_e(t)|e\rangle \;\right]$$

$$0 + c_g(t)\mathcal{H}_1|g\rangle + \hbar\omega_0 c_e(t)|e\rangle + c_e(t)\mathcal{H}_1|e\rangle =$$

$$i\hbar\frac{d}{dt}\left(c_g(t)|g\rangle + c_e(t)|e\rangle\right)$$

multiplying by $\langle g|$ and $\langle e|$:

$$i\hbar\frac{d}{dt}c_g(t) = c_e(t)\langle g|\mathcal{H}_1|e\rangle + 0$$

$$i\hbar\frac{d}{dt}c_e(t) = c_g(t)\langle e|\mathcal{H}_1|g\rangle + \hbar\omega_0 c_e(t)$$

these 2 eqns. are <u>coupled</u> by

$$\langle g|\mathcal{H}_1|e\rangle = \langle e|\mathcal{H}_1|g\rangle^*$$

we assume $\langle e|\mathcal{H}_1|e\rangle = \langle g|\mathcal{H}_1|g\rangle = 0$ no loss of generality note that whenever $\mathcal{H}_1(t) = 0$

$$c_g(t) = b_g, \quad \text{and} \quad c_e(t) = b_e\, e^{-i\omega_0 t}$$

where b_g, b_e are constants

If $\mathcal{H}_1$ is "small" in the sense that its matrix elements are small compared to those of $\mathcal{H}_0$, it is convenient (but general) to write

$$c_g(t) = b_g(t) \quad ; \quad c_e(t) = b_e(t)\, e^{-i\omega_0 t}$$

substituting these into the equations for $c_i(t)$:

$$i\hbar \frac{d}{dt} b_g(t) = b_e(t)\, e^{-i\omega_0 t} \langle g|\mathcal{H}_1|e\rangle$$

$$i\hbar \frac{d}{dt} b_e(t)\, e^{-i\omega_0 t} + \cancel{\hbar\omega_0 b_e(t)\, e^{-i\omega_0 t}} = \cancel{\hbar\omega_0 b_e(t)\, e^{-i\omega_0 t}} + b_g(t) \langle e|\mathcal{H}_1|g\rangle$$

simplifying and multiplying the 2^{nd} eq. by $e^{i\omega_0 t}$

$$i\hbar \dot{b}_g(t) = b_e(t)\, e^{-i\omega_0 t} \langle g|\mathcal{H}_1|e\rangle$$

$$i\hbar \dot{b}_e(t) = b_g(t)\, e^{i\omega_0 t} \langle e|\mathcal{H}_1|g\rangle$$

(still exact; now make some approximations) let $b_g(0) = 1$, $b_e(0) = 0$. let t be small enough that $|b_e(t)| \ll 1$. Then $b_g(t) \approx 1$ and we have

$$i\hbar \dot{b}_e(t) = e^{i\omega_0 t} \langle e|\mathcal{H}_1|g\rangle \qquad \text{for small } t$$

integrating

$$b_e(t) = \frac{1}{i\hbar}\int_0^t e^{i\omega_0 t} \langle e|\mathcal{H}_1|g\rangle \, dt$$

Note: this is like the F.T. of the perturbation

or

$$P_{g\to e} \equiv |b_e(t)|^2 = \frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_0 t}\langle e|\mathcal{H}_1|g\rangle dt\right|^2$$

Now we go to a case of particular interest in atomic physics, where $\mathcal{H}_1(t)$ is sinusoidal:

$$\mathcal{H}_1(t) = \hat{V}\sin\omega t = \frac{\hat{V}}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) \qquad \omega \geq 0$$

where $\hat{V}$ is time independent, hermitian operator with only off diagonal matrix elements

$$\langle e|\hat{V}|g\rangle = V_{eg} = V_{ge}^*$$

subsituting this form of $\mathcal{H}_1$ into $P_{g\to e}$:

$$P_{g\to e} = \frac{|V_{eg}|^2}{4\hbar^2}\left|\int_0^t \left[e^{i(\omega_0+\omega)t} - e^{i(\omega_0-\omega)t}\right]dt\right|^2 =$$

$$\frac{|V_{eg}|^2}{4\hbar^2}\left[\left(\frac{e^{i(\omega_0+\omega)t}}{i(\omega_0+\omega)} - \frac{e^{i(\omega_0-\omega)t}}{i(\omega_0-\omega)}\right)\Bigg|_0^t\right]^2 =$$

$$\frac{|V_{eg}|^2}{4\hbar^2}\left|\frac{1-e^{i(\omega_0+\omega)t}}{i(\omega_0+\omega)} - \frac{1-e^{i(\omega_0-\omega)t}}{i(\omega_0-\omega)}\right|^2$$

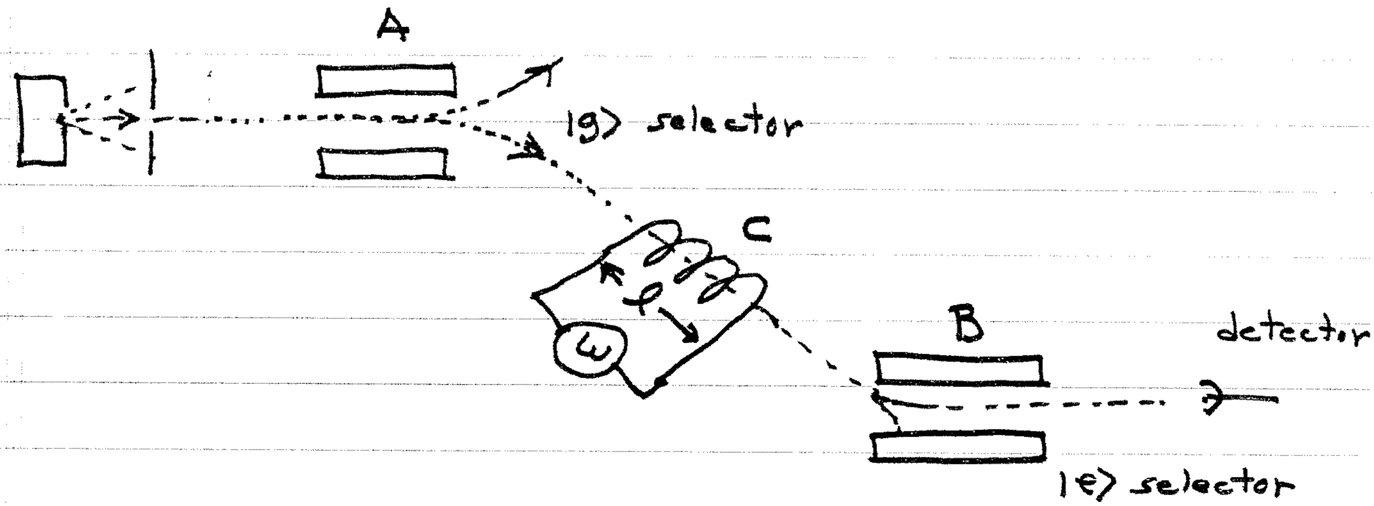
if we take $|\omega_0-\omega| \ll \omega_0, \omega_0+\omega$
(the resonant, or rotating wave approximation)
we can neglect the first term above.

re-writing $\dfrac{1-e^{i(\omega_0-\omega)t}}{\omega_0-\omega} = -i\,e^{i(\omega_0-\omega)t/2}\,\dfrac{\sin(\omega_0-\omega)t/2}{(\omega_0-\omega)/2}$

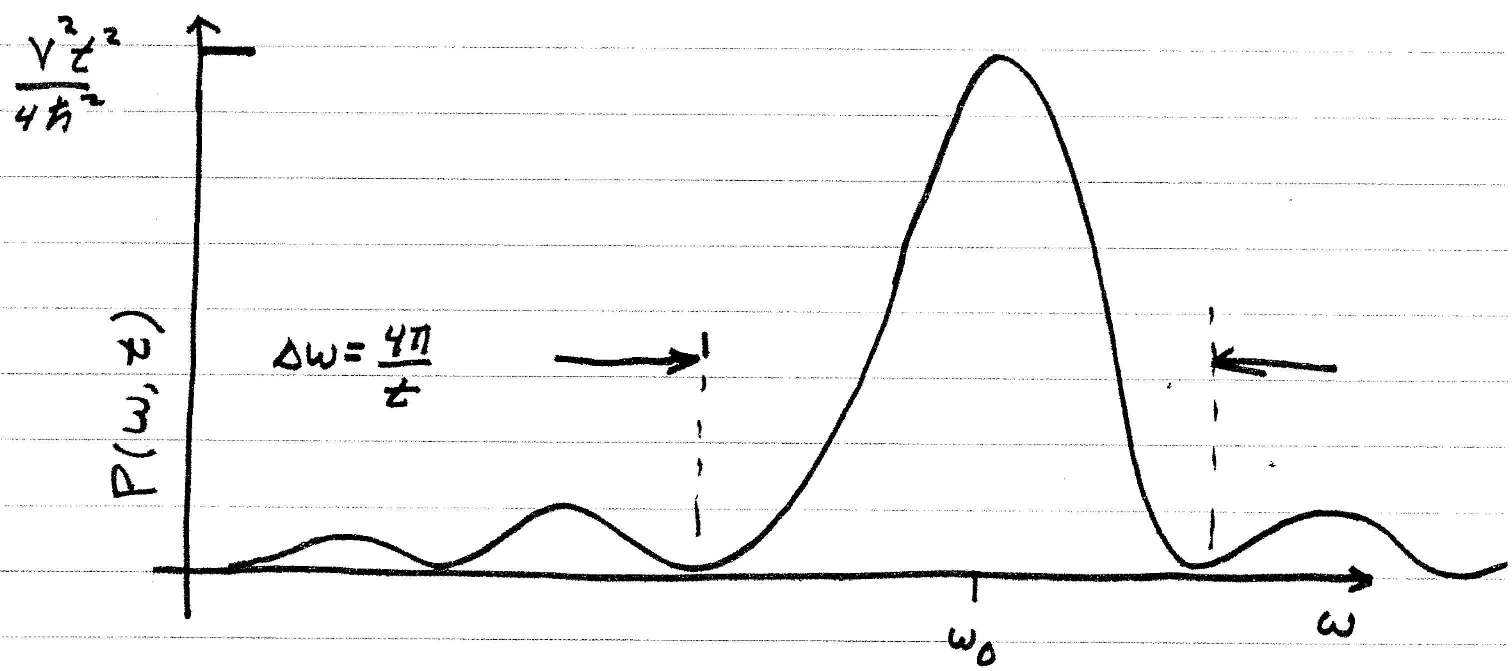
we have:

$$P_{g\to e}(t) = \frac{|V_{eg}|^2}{4\hbar^2}\,\frac{\sin^2(\omega_0-\omega)t/2}{[(\omega_0-\omega)/2]^2}$$

This is called the Rabi resonance formula
(I.I. Rabi invented magnetic resonance for atomic beams, Columbia University)



interaction time $t = \ell/v$



width goes like $1/t$, as in $\Delta E \Delta t \sim \hbar$

as $t \to \infty$, $\Delta\omega \to 0$, $\left(\text{but with } \dfrac{V^2 t^2}{4\hbar^2} \ll 1\right)$

recall the relation:

$$\lim_{\epsilon \to 0^+} \frac{\epsilon}{\pi} \frac{\sin^2 x/\epsilon}{x^2} = \delta(x)$$

with $x = (\omega_0 - \omega)/2$ and $t = 1/\epsilon$ we have

$$\lim_{t \to \infty} \frac{\sin^2[(\omega_0-\omega)t/2]}{[(\omega_0-\omega)/2]^2} = \pi t\, \delta([\omega_0-\omega]/2)$$

$$= 2\pi t\, \delta(\omega_0 - \omega)$$

so, substituting into $P_{g\to e}(t)$ from page 5:

see Cohen-Tannoudji et al, Quant. Mech. p 1301 for a careful treatment

$$P_{g\to e}(t) = \frac{\pi}{2\hbar^2} |V_{eg}|^2 \delta(\omega_0 - \omega)\, t$$

that is, the probability of having made a transition grows linearly with time — it is a rate process with

$$R = \frac{\pi}{2\hbar^2} |V_{eg}|^2 \delta(\omega_0 - \omega) \qquad \text{Fermi's Golden Rule}$$

If we write this in terms of energy: $E = \hbar\omega$, we will have:

$R = \frac{\pi}{2\hbar} |V_{eg}|^2 \delta(E_0 - E)$. But Fermi's golden rule is often written as: $R = \frac{2\pi}{\hbar} |V_{eg}|^2 \delta(E_0 - E)$ – why? If the perturbation is static ($\omega = 0$) both resonant and non-resonant terms contribute, giving a factor of 2 in the matrix element, a factor of 4 in the rate. Some people define $\mathcal{H}_1(t) = 2\hat{V}\sin\omega t$, and the extra factor of 2 recovers the familiar, static form of F.G.R.

Discussion:

How did a process going as t^2 on resonance become a rate process going as t?

What is the meaning of $t \to \infty$ while $\frac{|V_{eg}|^2 t^2}{4\hbar^2} \ll 1$?

note that $\frac{\sin^2 (\omega_0-\omega) t/2}{[(\omega_0-\omega)/2]^2}$ gets higher as t^{+2} but narrower as t^{-1}

so the area goes like t.

If the perturbation is truly monochromatic – a single value of $(\omega_0-\omega)$ – then the population grows as t^2 for short times.

But, if there is a spread of $(\omega_0-\omega)$, and <u>if</u> we can think of the different $(\omega_0-\omega)$ components contributing <u>incoherently</u> (with random phases) to the transition probability, we will need to sum (integrate) over all $(\omega_0-\omega)$.

If the breadth of $P_{g\to e}(\omega, t)$ is wider than the spread of $(\omega_0-\omega)$, then the integral goes as t and it is a rate!

The spread in $(\omega_0 - \omega)$ can come from a spread in ω (broad-band source) or in ω_0 (excited state not a true eigenstate)

If the excitation time is longer than $1/\Delta\omega$ and if during $1/\Delta\omega$ the excitation probability is low, we have a rate process

Otherwise, we may need to treat the problem coherently, which we will do later

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Now, having reviewed the Q.M. approach to rate processes, we will look at Einstein's 1917 treatment of radiative processes including absorption $(|g\rangle \rightarrow |e\rangle)$ and emission $(|e\rangle \rightarrow |g\rangle)$ both spontaneous and stimulated.

In reviewing Fermi's G.R. we saw the rate was governed by matrix elements $V_{ge} = \langle g|\hat{V}|e\rangle$.

We shall see that for optical, electric dipole transitions

$$\hat{V} = \vec{E} \cdot \hat{d}$$ where $\vec{E}$ is the classical electric field of the light and $\hat{d}$ is the dipole moment operator, $e\hat{r}$.

Einstein considered a 2-level (2-state) atom and a thermal (i.e., broad-band) radiation field. For a broad excitation

$$\mathcal{R}_{FGR} = \frac{\pi}{2\hbar^2} |d_{eg}|^2 \int E^2(\omega)\,\delta(\omega_0-\omega)\,d\omega$$

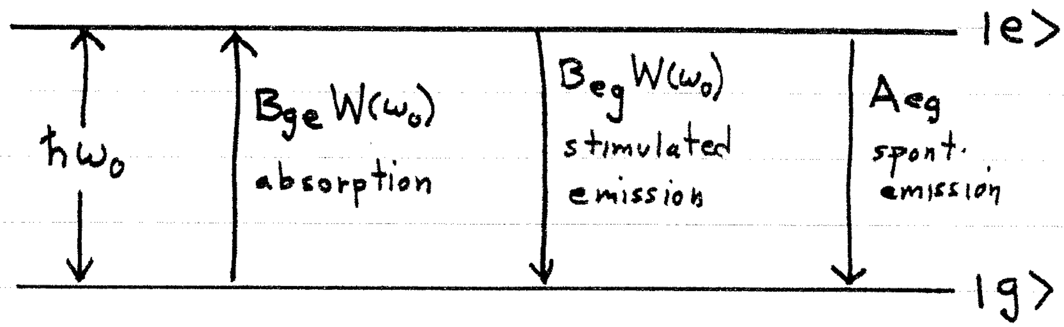
$$= \frac{\pi}{2\hbar^2} |d_{eg}|^2 \, E^2(\omega_0)$$

where $E^2(\omega_0)$ is the proportional to the E-M energy density (energy per unit volume, per unit frequency interval).

Without having this quantum result, Einstein wrote

$$\mathcal{R}_{ge} = B_{ge}\, W(\omega)$$ where $W(\omega)$ is the energy density per unit $dV d\omega$.

In addition, Einstein considered spontaneous emission (emission in the absence of light) <u>and</u> stimulated emission — induced by the presence of light. These latter two are $e \rightarrow g$ processes.



Each of these terms: $B_{ge}W(\omega_0)$, $B_{eg}W(\omega_0)$, and A_{eg} are rates per unit population in the state from which the transition starts.

Here, we place no restriction on the populations

$$N_e + N_g = 1$$

$$\dot{N}_g = -N_g B_{ge} W(\omega_0) + N_e B_{eg} W(\omega_0) + N_e A_{eg} = -\dot{N}_e$$

in equilibrium $\dot{N}_g = \dot{N}_e = 0$

so $$N_e A_{eg} = (N_g B_{ge} - N_e B_{eg}) W(\omega_0)$$

solve for $$W(\omega_0) = \frac{N_e A_{eg}}{N_g B_{ge} - N_e B_{eg}} = \frac{A_{eg}}{\frac{N_g}{N_e} B_{ge} - B_{eg}}$$

Now, let us assume that $W(\omega_0)$ is the thermal or "blackbody" or Planck distribution.

Then, we expect the atomic level populations to be in thermal equilibrium, according to the Boltzmann distribution:

$$\frac{N_g}{N_e} = e^{\hbar\omega_o/kT} \quad , \quad \text{so}$$

$$W_{thermal}(\omega_o) = \frac{A_{eg}}{\left(e^{\hbar\omega_o/kT}\right)B_{ge} - B_{eg}}$$

But, we know the Planck distribution to be:

$$W_{Planck}(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\,\frac{1}{e^{\hbar\omega/kT} - 1}$$

equating $W_{thermal}(\omega_o) = W_{Planck}(\omega_o)$:

$$\frac{A_{eg}}{e^{\hbar\omega_o/kT}B_{ge} - B_{eg}} = \frac{\hbar\omega^3/\pi^2c^3}{e^{\hbar\omega_o/kT} - 1}$$

This will be true only if: $B_{eg} = B_{ge}$ <u>and</u>

$$A_{eg} = \frac{\hbar\omega^3}{\pi^2c^3}\,B_{eg}$$

Discussion:

- We expected $B_{eg} = B_{ge}$ because Fermi's golden rule would work either direction and $|V_{eg}| = |V_{ge}|$ – but Einstein deduced this from statistical thermodynamics.

- We will calculate A_{eg}/B_{eg} from Q.M., but Einstein got it from thermal physics.

- Stimulated emission was unknown before Einstein; thermodynamics demands it; it is very important

- Why is there spontaneous emission? Einstein knew it happened, but without field quantization it appears not to exist.

Note: • many derivations of Einstein A + B coefficients include degeneracy factors for ground and excited levels – a common situation, but one we have ignored.

• the derivation depends on very little. It does not depend on the dipole approx, e.g.

Resonant Absorption Cross-Section

From the Einstein relationship of A + B, we can determine the cross section for absorption of light by a 2-level atom.

incoming flux, $\mathcal{F}$: # of particles per unit area, per sec.

scatterer with effective area σ (cross section)

absorption rate = $\mathcal{F} \cdot \sigma$ but we also know that

rate = $B_{ge} W(\omega_0)$ by combining these, we will get σ.

$\mathcal{F}$ = number of photons / area - seconds
(we will assume monochromatic at ω_0)

$W(\omega_0)$ = energy / volume - frequency interval
(assumed broadband)

we need to convert $\mathcal{F}$ into a $W(\omega_0)$.

A major difficulty is converting a monochromatic, beam-like flux to a broad-band isotropic thermal-like field as in Einstein's treatment.

Note that for a 2-state system with spontaneous emission from $|e\rangle \to |g\rangle$ as the only relaxation of $|e\rangle$, the width of the state $|e\rangle$ is A_{eg}. (More on this later in the ~~course~~ course.)

so:

$$\underbrace{W(\omega_0)}_{\substack{\text{energy per volume}\\ \text{per frequency}\\ \text{interval}}} \sim \frac{\overset{\text{energy per photon}}{\hbar\omega_0}\ \overset{\text{photons /area-s}}{\tilde{\mathcal{F}}}}{\underset{\text{velocity of light}}{c}\ \underset{\text{frequency bandwidth}}{A_{eg}}}$$

and $$\mathcal{F} \sim \frac{W c A_{eg}}{\hbar\omega_0} \quad ; \quad R_{g\to e} = \tilde{\mathcal{F}}\sigma = B_{ge} W$$

$$\sigma = \frac{BW}{\mathcal{F}} = \frac{BW\hbar\omega_0}{WA_{eg}c} = \frac{B}{A}\frac{\hbar\omega_0}{c} = \frac{\pi^2 c^2}{\omega_0^2} = \frac{\lambda^2}{4}$$

the details will depend on polarization and angular momentum, and we can do an exact QM calculation, but σ, on resonance is dependent

<u>only</u> <u>on</u> <u>λ^2</u>.

Appendix: the Planck radiation formula
(following Loudon)

Consider a single mode, freq. ω, of the E-M field in thermal equilibrium. The probability that it is excited to the n^{th} excited state is:

$$P(n) = \frac{e^{-n\hbar\omega/kT}}{\sum_{m=0}^{\infty} e^{-m\hbar\omega/kT}}$$

here we set the energy of $n=0$ to 0.

think of this as the probability of finding n photons in the mode if you measured

changing variables to define $U = e^{-\hbar\omega/kT}$,

$$P(n) = \frac{U^n}{\sum_{m=0}^{\infty} U^m}, \quad \text{but} \quad \sum_m U^m = \frac{1}{1-U}$$

so $P(n) = U^n(1-U)$

$$\langle n \rangle \equiv \sum_m P(m)\, m = (1-U)\sum_m m\, U^m = (1-U)\, U \frac{\partial}{\partial U}\sum_m U^m$$

$$\langle n \rangle = \frac{U}{1-U} = \frac{e^{-\hbar\omega/kT}}{1-e^{-\hbar\omega/kT}} = \frac{1}{e^{\hbar\omega/kT}-1}$$

multiplying by the mode density: ω^2/π^2c^3 and the energy $\hbar\omega$

gives $W_{Planck} = \dfrac{\hbar\omega^3}{\pi^2c^3}\,\dfrac{1}{e^{\hbar\omega/kT}-1}$