Using $\omega = ck$ $\quad \rho(\omega)\,d\omega = \frac{\omega^2\,d\omega}{\pi^2 c^3}$

Quantization of Field

One cannot derive quantum mechanics from classical mechanics, but casting the classical E+M problem as a harmonic oscillator will suggest how to do it: (based on our experience with the S.H.O)

$$\vec{E}_\perp(\vec{r},t) = \sum_{k\lambda} \hat{\epsilon}_{k\lambda}\left(E_{k\lambda}(t)\,e^{i\vec{k}\cdot\vec{r}} + E^*_{k\lambda}(t)\,e^{-i\vec{k}\cdot\vec{r}}\right)$$

or

$$\vec{A}(\vec{r},t) = \sum_{k\lambda} \hat{\epsilon}_{k\lambda}\left(A_{k\lambda}(t)\,e^{i\vec{k}\cdot\vec{r}} + A^*_{k\lambda}(t)\,e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\left.\begin{array}{l} \omega^2 E_{k\lambda}(t) + \dfrac{\partial^2 E_{k\lambda}(t)}{\partial t^2} = 0 \\ \text{or} \\ \omega^2 A_{k\lambda}(t) + \dfrac{\partial^2 A_{k\lambda}(t)}{\partial t^2} = 0 \end{array}\right\} \text{reminds us of } \boxed{-\omega^2 X(t) = \frac{\partial^2 X(t)}{\partial t^2}}$$

$$E_{k\lambda} = i\omega A_{k\lambda}$$

How did we quantize this?

Review of Quantized S.H.O

Harmonic oscillator:
in the Schrodinger picture,

$\hat{x}$ is an operator (time independent)
$\hat{p}$ is an operator " corresponding to $\vec{p} = \frac{d\vec{x}}{dt}$

Using $H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$, determined that

$$H = \hbar\omega (a^\dagger a + 1/2)$$

$$\hat{x} = a_0 (\hat{a} + \hat{a}^\dagger) \qquad a_0 = \sqrt{\frac{\hbar}{m\omega}} \rightarrow \text{Quantum unit of size for the S.H.O.}$$

$$\hat{p} = i m\omega a_0 (\hat{a}^\dagger - \hat{a}) \qquad m\omega a_0 \rightarrow \text{Quantum unit of momentum.}$$

Eigenstates $|n\rangle$, with time dependence $e^{i\omega n t}$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle \qquad [\hat{a}, \hat{a}^\dagger] = 1$$
$$\hat{a} |n\rangle = \sqrt{n}\, |n-1\rangle$$

Also, Since x, p are always 90° out of phase (classically thinking)
you can define quadrature operators based on normalized $\hat{x}$ + $\hat{p}$:

$$\hat{X} = \frac{1}{2} \hat{x}/a_0 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$$
$$\hat{Y} = \frac{1}{2} \hat{P}/m\omega a_0 = \frac{1}{2} i (\hat{a}^\dagger - \hat{a})$$

} operators that measure quadrature phase of state

$$\hat{a} = \hat{X} + i\hat{Y} \qquad \hat{a}^\dagger = \hat{X} - i\hat{Y}$$

note:
$$\langle n|\hat{X}|n\rangle = \langle n|\hat{Y}|n\rangle = 0$$
but a coherent state will have a phase.

For E + M fields in a specific mode

$$E_{k\lambda} = -\frac{\partial A_{k\lambda}}{\partial t}$$, draw the analogy
$\vec{A} \sim \vec{X}$
$\vec{E}_\perp \sim \vec{P}$ $\quad \vec{E}_{k\lambda} = i\omega \vec{A}_{k\lambda}$

$$\hat{x} \propto (\hat{a} + \hat{a}^\dagger), \Rightarrow \hat{A}_{k\lambda} \propto \left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} + \hat{a}^\dagger_{k\lambda} e^{i\vec{k}\cdot\vec{r}}\right)$$

Try:
$$\boxed{\begin{aligned} \hat{A} &= \sum_{k\lambda} A_{0k}\, \vec{\epsilon}_{k\lambda} \left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} + \hat{a}^\dagger_{k\lambda} e^{-i\vec{k}\cdot\vec{r}}\right) \\ \hat{E}_\perp &= \sum_{k\lambda} i E_{0k}\, \vec{\epsilon}_{k\lambda} \left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} - \hat{a}^\dagger_{k\lambda} e^{-i\vec{k}\cdot\vec{r}}\right) \end{aligned}}$$

where A_{0k} and E_{0k} are the quantum units of Vector potential and electric field.
$\Rightarrow$ This choice seems to match experimental reality.

Using the form for the energy of the field and setting that equal to $\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$, one determines the quantum units A_{0k} and E_{0k}

$$\boxed{E_{rad} = \frac{1}{2}\int_{cavity} dV \left(\epsilon_0 \vec{E}_\perp \cdot \vec{E}_\perp + \frac{\vec{B}\cdot\vec{B}}{\mu_0}\right)}$$ $\Big\}$ energy in cavity
$$\boxed{= \sum_{k\lambda} \hbar\omega_k (a^\dagger_{k\lambda} a_{k\lambda} + \tfrac{1}{2})}$$

exercise

$$A_{0k} = \sqrt{\frac{\hbar}{2\epsilon_0 V \omega_k}} \qquad E_{0k} = \omega_k A_{0k} = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}$$

note that the unit of field depends on the volume of the box and the frequency of mode.

Smaller cavity $\Rightarrow$ larger field / photon.

The eigenstates are called Fock states or number states, and they are a product over the modes $(\vec{k} + \lambda)$

$$|n_{k_1 1}, n_{k_1 2}, n_{k_2 1}, n_{k_2 2}, \ldots n_{k_i 1}, n_{k_i 2}, \ldots\rangle$$

In the Schrodinger picture, the eigen- states evolve at $e^{i\sum_{k\lambda}\hbar n_{k\lambda}\omega t}$

In the Heisenberg picture, the operators have the time dependence: e.g.

$$\vec{A}(\vec{r},t) = \sum_{k\lambda} \vec{\epsilon}_{k\lambda} \left\{ \hat{A}_{k0}\left(\hat{a}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \hat{a}^{+} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right)\right.$$

time dependent.

Implications of Quantized field operator:

Number states, which are eigenstates, are not eigenstates of the field.

e.g. for S.H.O.

$$\hat{x}\,|n\rangle = a_0\left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right),$$

so x is not precisely defined. Another way to say this is

expectation value $\longrightarrow$ $\langle n|\hat{x}|n\rangle = 0$ $\quad\quad$ $\langle n|\hat{x}^2|n\rangle = (2n+1)a_0^2 \neq 0$

For eigenstate $|n\rangle$

For Fields; we have the strange fact that

$$\langle n|\hat{E}|n\rangle = 0\,, \quad \langle n|\hat{E}^2|n\rangle = \frac{\hbar\omega_k}{2\epsilon_0 V}(2n+1)$$

Vacuum Part

to be discussed later

the vacuum part does not show up in most observables, takes some care to be consistent.

Like harmonic oscillators, a coherent state, or Glauber state, is the "most" classical state. It, too has an imprecise electric field. We will discuss these issues in more detail later in the course.

What about the longitudinal field?

$$\nabla \cdot \vec{E}_L = \rho/\epsilon_0 \implies \vec{E}_L(\vec{r},t) = -\nabla\phi(\vec{r},t)$$

$$\phi(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

The potential ϕ, and the longitudinal field $\vec{E}_L$, are entirely specified by the charge distribution and are not independent degrees of freedom. ($\vec{E}_L$ can always be eliminated from the eqns.)

This leads us to describe charged (point) particles:

$$\rho(\vec{r}) = \sum_i q_i \, \delta(\vec{r}-\vec{r}_i)$$

$$\vec{j}(\vec{r}) = \sum_i q_i \, \vec{V}_i \, \delta(\vec{r}-\vec{r}_i)$$

The Hamiltonian for the particles will be

$$H = \underbrace{\sum_i \frac{1}{2} m_i \bar{V}_i^2}_{\text{Kinetic Energy}} + \underbrace{\sum_{i \neq j} \frac{1}{8\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}_{\text{Coulomb Energy.}}$$

(for the time-being ignore the intrinsic magnetic moment of the particles, and leave out divergent self energies.)

For a particle subject to E&M forces, then the momentum conjugate to position, e.g.

$$\vec{P}_i : \quad [r_{ix}, p_{ix}] = i\hbar, \ [r_{iy}, p_{iy}] = i\hbar \text{ etc.}$$

is <u>not</u> $\vec{P}_i \neq m\vec{V}_i$, but rather

$$\vec{P}_i = m\vec{v}_i + q_i \vec{A}(\vec{r}_i)$$

So

$$\boxed{H = \sum_i \frac{1}{2m_i} \underbrace{(\vec{P}_i - q_i \vec{A}(\vec{r}_i))^2} + \sum_{i \neq j} \frac{1}{8\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}$$

this describes the coupling between charged particles and the free field.

So the Hamiltonian (in a box) is

$$\boxed{\begin{aligned} H &= H_A + H_R + H_I \\ H_A &= \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i\neq j} \frac{q_i q_j}{8\pi\epsilon_0 |\vec{r}_i - \vec{r}_j|} \\ H_R &= \sum_{k\lambda} \hbar\omega_{k\lambda}\left(a^\dagger_{k\lambda} a_{k\lambda} + 1/2\right) \\ H_I &= -\sum_i \frac{q_i}{m_i} \vec{p}_i \cdot \vec{A}(\vec{r}_i) + \sum_i \frac{q_i^2}{2m_i} \vec{A}^2(\vec{r}_i) \end{aligned}}$$

(using $(\vec{p} - q\vec{A})^2 = \vec{p}^{\,2} - 2q\,\vec{p}\cdot\vec{A} + q^2\vec{A}^2$)

(There is also a direct magnetic coupling to the intrinsic moment of the particles

$$H_I' = \sum_i \vec{\mu}_i \cdot \vec{B}(\vec{r}_i) \Rightarrow \text{discuss later})$$

The Hilbert space for the system is a tensor product of the space for particles and for the photons.

This Hamiltonian can be written in terms of $\hat{r}_i$ (instead of $\hat{p}_i$) and $\vec{E}$ (instead of $\vec{A}$) and the polarization & Magnetisation density $P(\vec{r}), M(\vec{r})$ details later in the course.