Review of E + M:

Maxwell's Eqns (once known as Maxwell-Heavyside eqn's)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\frac{1}{\mu_0} \nabla \times \vec{B} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J} \qquad \nabla \cdot \vec{B} = 0$$

Split vector field into transverse + longitudinal parts (can be done for any vector field), e.g $\vec{E} = \vec{E}_T + \vec{E}_L$, $\nabla \cdot \vec{E}_T = 0$
$\nabla \times \vec{E}_L = 0$

Using this split, Maxwell's eqn's split into ~free-field parts and charge-dependent ($\vec{B}$ is all transverse)

$$\nabla \times \vec{E}_T = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \cdot \vec{E}_L = \rho/\epsilon_0$$

$$J_L = -\epsilon_0 \frac{\partial \vec{E}_L}{\partial t}$$

$$\underbrace{\begin{gathered} \frac{1}{\mu_0} \nabla \times \vec{B} = \epsilon_0 \frac{\partial \vec{E}_T}{\partial t} + \vec{J}_T \\ \nabla \cdot \vec{E}_T = 0 \\ \nabla \cdot \vec{B} = 0 \end{gathered}}_{\text{Transverse}} \qquad \underbrace{\nabla \cdot J_L = -\frac{\partial \rho}{\partial t}}_{\text{Electrostatic-like}}$$

We can define potentials $\vec{A}$ + ϕ such that

$$\vec{E}_T = -\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}, \quad E_L = -\nabla \phi$$

(this is for the choice of Coulomb gauge, $\nabla \cdot \vec{A} = 0$)

Look at transverse eqn.'s in free space ($\vec{J}=0$)

$\nabla\times$ (first eqn.) and $\frac{\partial}{\partial t}$ (second eqn.) gives ($\nabla\times\nabla\times\vec{V} = \nabla(\nabla\cdot\vec{V}) - \nabla^2\vec{V}$)

$$\nabla^2 \vec{E}_T = \frac{1}{c^2}\frac{\partial^2 E_T}{\partial t^2}, \quad \text{with constraint } \nabla\cdot\vec{E}_T = 0$$

similarly for $\vec{A}$ or $\vec{B}$.

$\vec{E}_T$, $\vec{B}$ + $\vec{A}$ all satisfy the wave eqn.

The wave eqn. can be analyzed by mode, where the wave mode depends on boundary conditions. (e.g. cavity shape/size (cube, cyllindrical, sphere etc.) and periodic vs. reflective boundary conditions)

For simplicity, take a box of size L on all sides, with periodic boundary conditions. (other choices possible, but harder) (~~also drop the T subscript~~)

$$\vec{E}_T = \sum_{k\lambda} \hat{e}_{k\lambda}\left(E_{k\lambda}(t)e^{i\vec{k}\cdot\vec{r}} + E^*_{k\lambda}(t)e^{-i\vec{k}\cdot\vec{r}}\right)$$

$\vec{k} \rightarrow$ wave vector

$\lambda \rightarrow$ polarization

decompose into plane-waves

$$k^2 E_{k\lambda}(t) + \frac{1}{c^2}\frac{\partial^2 E_{k\lambda}(t)}{\partial t^2} = 0, \quad \vec{k}\cdot\hat{e}_{k\lambda} = 0$$

soln. $E_{k\lambda}(t) = E_{k\lambda} e^{i\omega}$

$\omega = ck$

$\hat{e}_{k\lambda}$ is a unit vector, but the $\nabla\cdot\vec{E}_T$ constraint restricts the 3 possible directions to 2, perpendicular to $\vec{k}$ only.

Similar eqns. for $\vec{A}$ + $\vec{B}$, but since $\vec{B}$ + $\vec{E}_T$ are related by $\nabla\times\vec{E}_T = -\frac{\partial\vec{B}}{\partial t}$, paramaterize the polarization vector differently:

$$\vec{B} = \sum_{k\lambda} \frac{\vec{k} \times \hat{\epsilon}_{k\lambda}}{|k|} \left(\vec{B}_{k\lambda}(t)\, e^{i\vec{k}\cdot\vec{r}} + \vec{B}^*_{k}(t)\, e^{-i\vec{k}\cdot\vec{r}} \right)$$

$$\vec{A} = \sum_{k\lambda} \hat{\epsilon}_{k\lambda} \left(A_{k\lambda}(t)\, e^{i\vec{k}\cdot\vec{r}} + A^*_{k\lambda}(t)\, e^{-i\vec{k}\cdot\vec{r}} \right)$$

$E_{k\lambda}$, $B_{k\lambda}$, + $A_{k\lambda}$ are not independent, and only one is needed to specify the fields. (e.g. $E_{k\lambda}$ or $A_{k\lambda}$)

[easy to show $\Rightarrow i\vec{k} \times \hat{\epsilon}_{k\lambda} E_{k\lambda} = i\omega\left(\frac{\vec{k}}{|k|} \times \hat{\epsilon}_{k\lambda}\right) B_{k\lambda} \Rightarrow B_{k\lambda} = \frac{1}{c} E_{k\lambda}$]

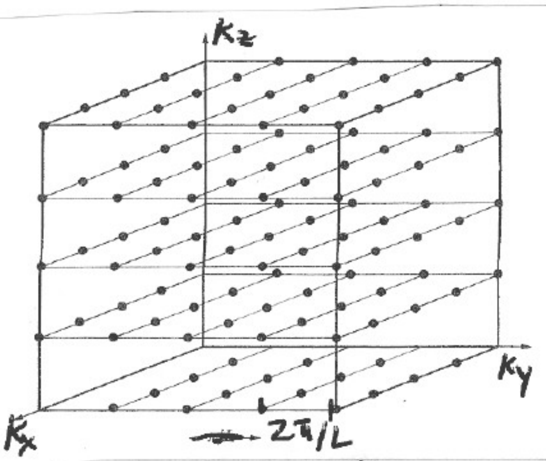
Before quantizing field, look at mode density.

How many modes in k-space are there at a given $|\vec{k}|$?

periodic boundary conditions on box $\Rightarrow$

$$k_z L = 2\pi n_z, \quad k_y L = 2\pi n_y, \quad k_x L = 2\pi n_x,$$

so within each k-space box $(\Delta k)^3$, $\Delta k = \frac{2\pi}{L}$, there is a mode.



The # of modes between k + $k+dk$ is

$$\underbrace{(4\pi k^2 dk)}_{\text{Volume of shell at radius } k} \underbrace{\left(\frac{L}{2\pi}\right)^3}_{\left(\frac{1}{\Delta k}\right)^3,\ \text{k-space mode density}} \times \underbrace{2}_{\text{polarization}}$$

$$\rho(k)\, dk = \frac{k^2 dk}{\pi^2}$$

, $\rho(k)$ is mode density per Volume, L^{-3}