HW #4

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1) a) steady state $\rightarrow$ time derivatives $= 0$

$$\left.\begin{aligned} 0 &= \delta v - \frac{\Gamma}{2} u \\ 0 &= -\delta u - \Omega w - \frac{\Gamma}{2} v \\ 0 &= \Omega v - \Gamma w - \frac{\Gamma}{2} \end{aligned}\right\} \longrightarrow \boxed{\begin{aligned} u &= \frac{2\delta\Omega}{4\delta^2+\Gamma^2+2\Omega^2} \\ v &= \frac{\Gamma\Omega}{4\delta^2+\Gamma^2+2\Omega^2} \\ w &= \frac{-\frac{1}{2}(4\delta^2+\Gamma^2)}{4\delta^2+\Gamma^2+2\Omega^2} \end{aligned}}$$

b)
$$\left.\begin{aligned} \sigma_{ee} + \sigma_{gg} &= 1 \\ w &= \tfrac{1}{2}(\sigma_{ee} - \sigma_{gg}) \end{aligned}\right\} \longrightarrow \boxed{\sigma_{ee} = w + \frac{1}{2} = \frac{\Omega^2}{4\delta^2+\Gamma^2+2\Omega^2}}$$

c)
$$\boxed{R_{spont.} = \Gamma \cdot \sigma_{ee} = \Gamma \frac{\Omega^2}{4\delta^2+\Gamma^2+2\Omega^2}}$$

d) Max R_{spont} clearly occurs on resonance $(\delta = 0)$ in a strong field $(\Omega >> \Gamma)$

$$\boxed{R_{max} = \frac{\Gamma\Omega^2}{\Gamma^2+2\Omega^2} \xrightarrow[\Omega >> \Gamma]{} \frac{\Gamma}{2}}$$

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2)

Fermi

$$R_F = \frac{1}{8}\frac{I}{I_s}\frac{\Gamma^3}{\delta^2+\frac{\Gamma^2}{4}}$$

Bloch

$$\boxed{R_B = \frac{\Gamma\Omega^2}{4\delta^2+\Gamma^2+2\Omega^2}} = \frac{\Gamma\left(\frac{\Omega}{\Gamma}\right)^2}{\left(\frac{2\delta}{\Gamma}\right)^2+1+2\left(\frac{\Omega}{\Gamma}\right)^2}$$

$$\Gamma = \frac{\omega^3 e^2 r_{eg}^2}{3\pi\varepsilon_0\hbar c^3}$$

$$\Omega = \frac{-eE}{2\hbar} r_{eg}, \qquad I = \frac{\varepsilon_0 c E^2}{2} \longrightarrow I = \frac{2\hbar^2\Omega^2\varepsilon_0 c}{e^2 r_{eg}^2}$$

$$I_s = \Gamma \frac{\hbar\omega^3}{3c^2\pi}$$

so
$$\frac{I}{I_s} = \frac{2\Omega^2}{\Gamma^2}$$

$$\boxed{R_F = \frac{\Gamma\Omega^2}{4\delta^2+\Gamma^2}} = \frac{\Gamma\left(\frac{\Omega}{\Gamma}\right)^2}{\left(\frac{2\delta}{\Gamma}\right)^2+1}$$

R_F can not saturate due to Ω $\rightarrow$ it was developed as a perturbation, and is thus not valid for large fields.

$R_F = R_B$ for $\Omega << \Gamma$, $R_F = R_{B_{max}} = \frac{\Gamma}{2}$ for $\delta = 0$, $\Omega = \frac{\Gamma}{\sqrt{2}}$

3) $H_{so} = \xi(r)\,\vec{l}\cdot\vec{s}$

a) $\vec{E} = -\frac{1}{e}\nabla U$

electron sees $\vec{B} = -\vec{v}\times\frac{\vec{E}}{c} = \frac{1}{ec}\vec{v}\times\nabla U$

Take $H_{so} = -\vec{\mu}\cdot\vec{B} = -\frac{g}{2}\frac{e}{m}\vec{s}\cdot\vec{B}$

Assume U to be spherically symmetric:

$$H_{so} = -\frac{g}{2}\frac{e}{mc}\vec{s}\cdot\left(\frac{1}{ec}\frac{\vec{p}}{m}\times\frac{\vec{r}}{|\vec{r}|}\frac{\partial U}{\partial r}\right)$$

$$= \frac{g}{2m^2c^2}\vec{s}\cdot\frac{(\vec{r}\times\vec{p})}{|\vec{r}|}\frac{\partial U}{\partial r}$$

$$= \frac{g}{2m^2c^2}\frac{1}{r}\frac{\partial U}{\partial r}\vec{L}\cdot\vec{S}$$

So $\boxed{\xi(r) = \frac{g}{2m^2c^2}\frac{1}{r}\frac{\partial U}{\partial r}}$

b) $\vec{w}_T \simeq \frac{1}{2}\frac{\vec{a}\times\vec{v}}{c^2}$

$$\vec{a} = \frac{\vec{F}}{m} = -\frac{1}{m}\nabla U = -\frac{1}{m}\frac{\vec{r}}{|\vec{r}|}\frac{\partial U}{\partial r}$$

so $\vec{a}\times\vec{v} = -\frac{1}{m}\frac{\partial U}{\partial r}\frac{1}{|\vec{r}|}\vec{r}\times\frac{\vec{p}}{m} = -\frac{1}{m^2}\frac{\partial U}{\partial r}\frac{1}{|r|}\vec{L}$

$$H_{so} = \frac{g}{2m^2c^2}\frac{1}{r}\frac{\partial U}{\partial r}\vec{L}\cdot\vec{S} + \frac{1}{2c^2}\left(-\frac{1}{m^2}\frac{1}{r}\frac{\partial U}{\partial r}\vec{L}\cdot\vec{S}\right)$$

$$= \frac{(g-1)}{2m^2c^2}\frac{1}{r}\frac{\partial U}{\partial r}\vec{L}\cdot\vec{S} \longrightarrow \boxed{\xi(r) = \frac{(g-1)}{2m^2c^2}\frac{1}{r}\frac{\partial U}{\partial r}}$$

c) Want $\langle 2p|H_{so}|2p\rangle = \langle 2p|\xi(r)\vec{L}\cdot\vec{S}|2p\rangle$

$l=1$, $s=\frac{1}{2}$

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\hbar^2\big(j(j+1)-s(s+1)-l(l+1)\big) = \frac{\hbar}{2}\begin{cases}1 & \text{for } j=\frac{3}{2}\\ -2 & \text{for } j=\frac{1}{2}\end{cases}$$

So $E_{so\,(j=\frac{3}{2})} - E_{so\,(j=\frac{1}{2})} = \left\langle 2p\left|\frac{g-1}{2m^2c^2}\frac{1}{r}\frac{\partial U}{\partial r}\right|2p\right\rangle\cdot\frac{1}{2}\hbar^2\big(1-(-2)\big)$

$$U = \frac{-e^2}{4\pi\varepsilon_0 r} \rightarrow \frac{1}{r}\frac{\partial U}{\partial r} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r^3}$$

$$\Delta E = \frac{g-1}{2m^2c^2}\frac{e^2}{4\pi\varepsilon_0}\left\langle 2p\left|\frac{1}{r^3}\right|2p\right\rangle\frac{3}{2}\hbar^2$$

$$\boxed{\begin{aligned}\Delta E &= \frac{g-1}{2m^2c^2}\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{24a_0^3}\cdot\frac{3}{2}\hbar^2\\ &\approx 4.5\times10^{-5}\ \text{eV}\end{aligned}}$$