Homework #8

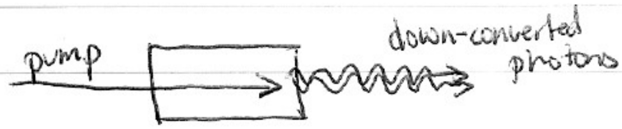
Consider the nonlinear process in a medium that leads to spontaneous down-conversion of a pump-photon into two photons. For simplicity, consider degenerate down conversion, where the two emitted photons are in the same mode.

This process has terms that look like

$$\hat{a}^{+}\hat{a}^{+}\hat{b}\ , \qquad \text{where} \quad [\hat{b},\hat{b}^{+}]=1, \quad [\hat{a},\hat{a}^{+}]=1$$

$\hat{b}^{+}$ → creation operator for pump-field

$\hat{a}^{+}$ → creation operator for down-converted photons.



ignore the spatial dependence $e^{i\vec{k}\cdot\vec{r}}$, i.e. take $\vec{r}=0$

a The interaction-free Hamiltonian is $H = \hbar\omega_p \hat{b}^{+}b + \frac{\hbar\omega_p}{2}\hat{a}^{+}\hat{a}$

In order to be Hermitian, the interaction Hamiltonian must include the other term $\hat{a}\hat{a}\hat{b}^{+}$:

$$H_I = V_0\left(\hat{a}^{+}\hat{a}^{+}\hat{b} + \hat{a}\hat{a}\hat{b}^{+}\right)$$

Treat the pump field classically, i.e. in a coherent state $|\beta\rangle$,

$$\hat{b}|\beta\rangle = \beta|\beta\rangle \qquad \beta = |\beta|e^{i\phi}$$

$$\hat{b}^{+}|\beta\rangle \simeq \beta^{*}|\beta\rangle \rightarrow \text{for } n_b \gg 1$$

a) For an initial state that has no photons in the $\hat{a}$ mode, (vacuum state $|0\rangle_a$) find the time evolution of the $|0\rangle_a$ state, or equivalently the time-evolution of $\hat{a}$, $\hat{a}^\dagger$:

Suggested approaches:

- In the Schrodinger picture, find the time-evolution operator, $U = e^{i\frac{Ht}{\hbar}}$, and apply it to $|0\rangle_a$

Or

- In the Heisenberg picture, $|0\rangle_a$ has no time dependence, but $\hat{a} = \hat{a}(t)$, and $\hat{a}^\dagger = \hat{a}^\dagger(t)$ do, (as well as all observables) and specifying the oservables & their expectation values specifies the time-dependent solutions:

$$i\hbar \frac{d\hat{a}}{dt} = [\hat{a}, H], \quad i\hbar \frac{d\hat{a}^\dagger}{dt} = [\hat{a}^\dagger, H]$$

The resulting state is "squeezed vacuum".

b) what is $\langle n \rangle$ after time t?

What is $\langle \Delta \hat{X}^2 \rangle$ after time t? $\quad \hat{X} = \frac{\hat{a} + \hat{a}^\dagger}{2}$

What is $\langle \Delta \hat{Y}^2 \rangle$ after time t? $\quad \hat{Y} = \frac{\hat{a} - \hat{a}^\dagger}{2i}$

$\Delta X^2 = \langle X^2 \rangle - \langle X \rangle^2$ etc.

what is $\langle X \rangle$?

what is $\langle Y \rangle$?

$\langle n^2 \rangle$?

c) Qualitatively describe the properties of the squeezd state after a "displacement" that adds a coherent part to it:

$$D(\alpha)\,|\text{squeezed vacuum}\rangle$$

$$D(\alpha) = e^{\alpha \hat{a}^{+} - \alpha^{*} \hat{a}} \qquad \alpha = |\alpha| e^{i\theta}$$

e.g. $\langle n \rangle$, $\langle n^2 \rangle$, $\langle X \rangle$, $\langle Y \rangle$ etc.