Homework #8

Consider the nonlinear process in a medium that leads to spontaneous down-conversion of a pump-photon into two photons. For simplicity, consider degenerate down conversion, where the two emitted photons are in the same mode.

This process has terms that look like

\[ \hat{a} \hat{a}^\dagger \hat{b} \], \hspace{1cm} \text{where} \hspace{1cm} \left[ \hat{b}^\dagger, \hat{b} \right] = 1 \hspace{1cm} \left[ \hat{a}, \hat{a}^\dagger \right] = 1 \]

\[ \hat{b}^\dagger \rightarrow \text{creation operator for pump field} \]

\[ \hat{a}^\dagger \rightarrow \text{creation operator for down-converted photons} \]

\[ \begin{array}{c}
\text{pump} \\
\downarrow \\
\text{down-converted} \\
\text{photons} \\
\end{array} \hspace{1cm} \text{ignore the spectral dependence} \hspace{1cm} e^{i\vec{k}_p \cdot \vec{r}} \hspace{1cm} \text{i.e. take} \hspace{1cm} \vec{r} = 0 \]

The interaction-free Hamiltonian is \[ H = \hbar \omega_p \hat{b}^\dagger \hat{b} + \frac{\hbar \omega_p \hat{a}^\dagger \hat{a}}{2} \]

In order to be Hermitian, the interaction Hamiltonian must include the other term \[ \hat{a} \hat{a}^\dagger \hat{b} \]:

\[ H_2 = \begin{array}{c}
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\text{V}_0 \end{array} \end{array} \\
\left( \hat{a} \hat{a}^\dagger \hat{b} + \hat{a} \hat{a} \hat{b}^\dagger \right) \\
\end{array} \]

Treat the pump field classically, i.e. in a coherent state \[ |\beta> \] such that:

\[ \hat{b} |\beta> = \beta |\beta> \hspace{1cm} |\beta> = |\beta| e^{i\phi} \]

\[ \hat{b}^\dagger |\beta> \approx \beta^* |\beta> \rightarrow \text{for} \hspace{1cm} N_b \gg 1 \]
a) For an initial state that has no photons in the $\hat{a}$ mode, (vacuum state $|0\rangle_a$) find the time evolution of the $|0\rangle_a$ state, or equivalently the time-evolution of $\hat{a}$, $\hat{a}^\dagger$.

Suggested approaches:
- In the Schrödinger picture, find the time-evolution operator, $U = e^{-i\mathcal{H}t/\hbar}$, and apply it to $|0\rangle_a$

  OR

- In the Heisenberg picture, $|0\rangle_a$ has no time dependence, but $\hat{a} = \hat{a}(t)$, and $\hat{a}^\dagger = \hat{a}^\dagger(t)$ do, (as well as all observables) and specifying the observables at their expectation values specifies the time-dependent solutions:

$$i\hbar \frac{d\hat{a}}{dt} = [\hat{a}, \mathcal{H}], \quad i\hbar \frac{d\hat{a}^\dagger}{dt} = [\hat{a}^\dagger, \mathcal{H}]$$

The resulting state is "squeezed vacuum".

b) What is $|n\rangle$ after time $t$?

What is $\langle \Delta \hat{x}^2 \rangle$ after time $t$? $\hat{x} = \hat{a} + \hat{a}^\dagger$

What is $\langle \Delta \hat{y}^2 \rangle$ after time $t$? $\hat{y} = \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i}$

$\Delta x^2 = \langle x^2 \rangle - \langle x \rangle^2$ etc.

What is $\langle x \rangle$?

What is $\langle y \rangle$?

$\langle n^2 \rangle$?
c) Qualitatively describe the properties of the state after a "displacement" that adds a coherent part to it.

\[ D(\alpha) | \text{squeezed vacuum} > \]

\[ D(\alpha) = e^{\alpha \hat{a}^+ - \alpha^* \hat{a}} \quad \alpha = |\alpha| e^{i\theta} \]

e.g. \( \langle n \rangle \), \( \langle n^2 \rangle \), \( \langle x \rangle \), \( \langle y \rangle \) etc.