October 23,2003

Physics 721 Mid-term exam

The rules:

You are free to consult any *non-human* resource in order to complete this exam.

Due at noon on October 28, 2003 in room 2127 3301 Physics.

NO EXCEPTIONS.

Question 1:
The ground state of ^{23}Na is 3s $^2S_{1/2}$, its nuclear spin is I=3/2, and its ground-state hyperfine splitting 1.77 GHz.

a) Make an energy level diagram of the ground state. Explain the origin of the quantum number, typically called F, with z-projection M_F, used to label each level, and give the values of F and M_F for each level in your diagram. (Assume that a weak static magnetic field provides the quantization axis.)

b) For a sample in thermal equilibrium at room temperature, calculate what fraction of the population of the 3s $^2S_{1/2}$ ground state has the largest value of F.

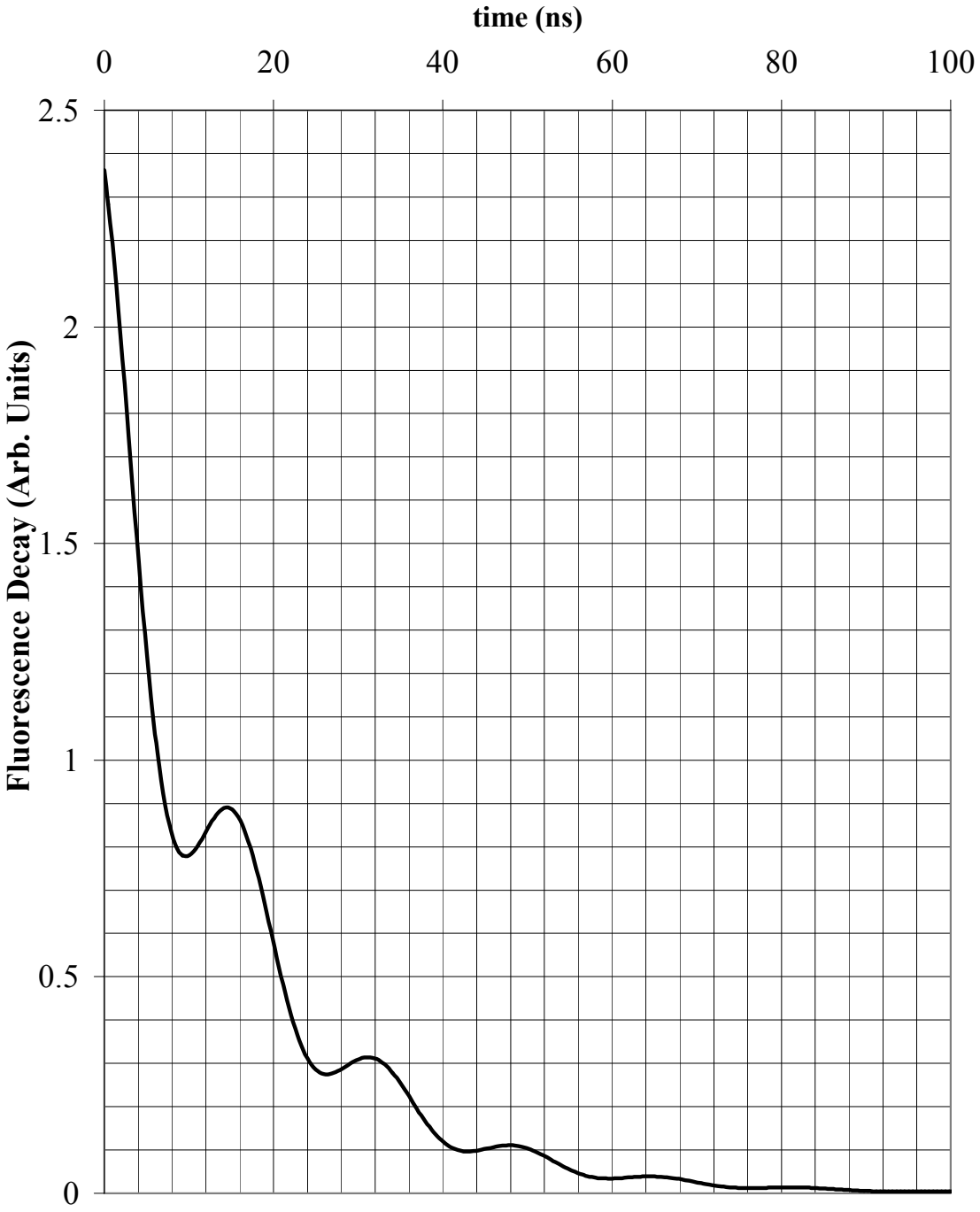
c) The D_2 line connects the 3s $^2S_{1/2}$ ground state with the 3p $^2P_{3/2}$ excited state. Make an energy level diagram of the 3p $^2P_{3/2}$ state; label each level with its quantum numbers F and M_F.

d) For electric dipole transitions involving the D_2 line, state the appropriate selection rules for changes in quantum numbers F and M_F. Assume the direction of propagation of the laser is parallel to the weak magnetic field. State explicitly the polarization of the light.

e) The separation of the energy levels in the 3p $^2P_{3/2}$ manifold from the Hyperfine Interaction are F=3 to F=2: 59.6 MHz, F=2 to F=1: 35.5 MHz, F=1 to F=0: 16.5 MHz. Find the magnetic dipole hyperfine constant A and electric quadrupole hyperfine constant B for this state in ^{23}Na.

f) In the presence of a magnetic field the 3p $^2P_{3/2}$ excited state manifold splits again according to its M_F quantum number. Find the magnetic field necessary to encounter the first "crossing" of the lowest state in the F=3 manifold with the upper state of the F=2 manifold.

g) When measuring the lifetime of the 3p $^2P_{3/2}$ excited state the following decay was observed. From the attached plot estimate the lifetime and the frequency or frequencies of quantum beats.


time (ns)
0
20
40
60
80
100
Fluorescence Decay (Arb. Units)
2.5
2
1.5
1
0.5
0

Question 2:
The Coherent state is a minimum uncertainty state with equal variances in its two quadratures. It can be generated from the vacuum by the Displacement Operator $D(\alpha)$ in such a way that:
$|\alpha\rangle = D(\alpha)|0\rangle$, where $D(\alpha) = \exp(\alpha a^{+} - \alpha^{*} a)$

a) Find the expectation value of the number of photons in a coherent state $|\alpha\rangle$.
b) Calculate the variance in the photon number for a coherent state $|\alpha\rangle$
c) Calculate the probability distribution as a function of photon number P(n) for a coherent state $|\alpha\rangle$
d) Plot the probability distribution of a coherent state with mean photon number 40.

Squeezed states, a generalization of Coherent States, of the electromagnetic field are a class of minimum uncertainty states whose uncertainty can be less than the standard quantum limit in one of the quadratures while the other will be larger as to maintain the limit on their product given by the uncertainty principle.

The states are produced by the Squeezing operator $S(\xi)$

$S(\xi) = \exp[1/2(\xi^{*} a^{2} - \xi a^{+2})]$, where $\xi = re^{i\theta}$

Then the squeezed vacuum is:
$|\xi\rangle = S(\xi)|0\rangle$

The operator has the following properties:
$S^{+}(\xi) = S^{-1}(\xi) = S(-\xi)$

The action of the operator on the creation an anhiliation operator is to "rotate" the basis as:

$S^{+}(\xi)\, a\, S(\xi) = a \cosh r - a^{+} e^{i2\theta} \sinh r = b(\xi)$
$S^{+}(\xi)\, a^{+} S(\xi) = a \cosh r - a^{+} e^{-i2\theta} \sinh r = b^{+}(\xi)$

e) Calculate the expectation value of the number of photons ($a^{+}a$) in a squeezed vacuum.
f) Calculate the variance in the photon number for the squeezed vacuum.

It is possible to add amplitude (α) to a Squeezed state using the displacement operator such that it has an arbitrary average photon number. Then the displaced squeezed state

$D(\alpha)S(\xi)|0\rangle = |\alpha,\xi\rangle$

g) Find the photon number distribution for a displaced coherent state $|\alpha,\xi\rangle$,
and plot the value for the case of $\alpha = 40$ and $\xi = 10$.

Question 3:

Consider a two-level atom with states |g> and |e> and energy difference $\hbar\omega_0$, coupled by a laser with an atom-laser coupling strength given by the Rabi frequency Ω. The time-dependent electric field is:

$$E = E_0 \cos(\omega_0 t + \alpha\, t^2)\,.$$

a) Find the time-dependent eigenvalues and eigenstates.

b) Plot the eigenvalues vs. t from $-\tau < t < \tau$, assuming $\alpha = \Delta\omega/\tau,\ \Delta\omega = 5\Omega$.

c) If the laser is turned on suddenly at $t = -\tau$, and the atom was initially in state |g>, how much population is in the upper dressed state?

d) Assuming the atom starts in the lower dressed state at $t = -\tau$, evaluate the adiabaticity parameter $\eta(t) = \hbar\langle\varphi_2 \mid \dot{\varphi}_1\rangle/(E_1 - E_2)$ at t=0. (Hint: A series expansion around t=0 will make the algebra easier).

e) If $\tau = 0.1\ \Omega^{-1}$, is the population predominantly in |g> or |e> at $t = \tau$?

Question 4:

Consider a J=1 -> J=1 transition in an atom, illuminated by a laser beam with $\vec{k} = k\hat{z}$ and polarization σ_+, and a laser beam with $\vec{k} = -k\hat{z}$ and polarization σ_-. This is a system that will exhibit coherent population trapping as discussed in class. Now we will include the external degrees of freedom in the problem.

a) Show why we can consider this a three-level Λ system only including the m=1, and m=–1 ground states and the m=0 excited state.

b) We can write the atom-laser interaction as:
$$V_{AL} = \frac{\hbar\Omega}{2}\left[-\frac{1}{\sqrt{2}}e^{ik\hat{z}}|e\rangle\langle -1| + \frac{1}{\sqrt{2}}e^{-ik\hat{z}}|e\rangle\langle 1|\right]e^{-i\omega_L t} + h.c.$$
Considering the external degrees of freedom, and using the fact that $e^{ik\hat{z}}$ is a translation in momentum space, find the non-coupled state $|\Psi_{nc}(p)\rangle$ in terms of basis states $|i,p\rangle$ where *i=1,-1,or e*, and *p* is the momentum, and show that $V_{AL}|\Psi_{nc}(p)\rangle = 0$.

c) Find the coupled state in this basis, and evaluate $V_{AL}|\Psi_c(p)\rangle$.

d) The total Hamiltonian for the atom includes a term, $H_a = \frac{\hat{P}^2}{2m}$, where $\hat{P}$ is the momentum operator along *z*. Evaluate $H_a|\Psi_{nc}(p)\rangle$ and $H_a|\Psi_c(p)\rangle$ and express the results in the $|\Psi_c(p)\rangle, |\Psi_{nc}(p)\rangle$ basis.

e) Evaluate $\langle\Psi_c(p)|H_a|\Psi_{nc}(p)\rangle$. Discuss the evolution of a non-coupled state with *p*=0, and with $p \neq 0$. This result is the basis for velocity selective coherent population trapping (VSCPT).