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These basis set is the polarization set. it can be linear or elliptical in general. The particular basis for a problem can be best determined wst/ the end or the most convenient moment to simplify the calculation.

Then

$$\vec{A}(\vec{r}, t) = \frac{1}{\epsilon_0^{1/2} L^{3/2}} \sum_s \sum_{k, s} [C_{ks} \vec{E}_{ks} e^{-i\omega t} + C_{-ks}^* \vec{E}_{-ks}^* e^{i\omega t}] e^{i\vec{k} \cdot \vec{r}}$$

[now changing the index]

$$= \frac{1}{\epsilon_0^{1/2} L^{3/2}} \sum_s \sum_{k, s} [C_{ks} \vec{E}_{ks} e^{i(k \cdot \vec{r} - \omega t)} + C_{ks}^* \vec{E}_{ks}^* e^{-i(k \cdot \vec{r} - \omega t)}]$$

$$= \frac{1}{\epsilon_0^{1/2} L^{3/2}} \sum_s \sum_k [u_{ks}(t) \vec{E}_{ks} e^{i\vec{k} \cdot \vec{r}} + u_{ks}^*(t) \vec{E}_{ks}^* e^{-i\vec{k} \cdot \vec{r}}]$$

where $u_{ks}(t) = C_{ks} e^{-i\omega t}$

$\vec{E}_{ks} e^{i\vec{k} \cdot \vec{r}}$ are the fundamental mode functions with complex amplitude $u_{ks}(t)$

Each mode is labeled by a wave vector \vec{k} and polarization index s

The mode function satisfies the Helmholtz equation

$$(D^2 + k^2) E_{ks} e^{ik \cdot \vec{r}} = 0$$

The mode amplitude $u_{ks}(t)$ satisfies the H. D.

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2 \right) u_{ks}(t) = 0$$

Then the mode expansions of the electric and magnetic field are: from $\vec{E} = -\frac{\partial}{\partial t} \vec{A}$ and $\vec{B} = \nabla \times \vec{A}$

$$\vec{E}(\vec{r}, t) = \frac{i}{\epsilon_0'' L^{3/2}} \sum_k \sum_s \omega [u_{ks}(t)] \vec{e}_{ks} e^{i \vec{k} \cdot \vec{r}} - \text{c.c.}$$

$$\vec{B}(\vec{r}, t) = \frac{i}{\mu_0'' L^{3/2}} \sum_k \sum_s [u_{ks}(t)] (\vec{k} \times \vec{e}_{ks}) e^{i \vec{k} \cdot \vec{r}} - \text{c.c.}$$

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The energy of the EM field is:

$$H = \frac{1}{2} \int L^3 \left[\epsilon_0 \vec{E}^2(\vec{r}, t) + \frac{1}{\mu_0} \vec{B}^2(\vec{r}, t) \right] d^3 r$$

or in terms of the mode amplitudes

$$H = \frac{1}{2} \sum_k \sum_s \omega^2 |u_{ks}(t)|^2$$

remember
 $\int L^3 e^{i(\vec{k}-\vec{k}') \cdot \vec{r}} d^3 r = L^3 \delta_{\vec{k}\vec{k}'}$
and $(\vec{k} \times \vec{e}^*) \cdot (\vec{k} \times \vec{e}_{k'}) = k^2 \epsilon_{k'k} \cdot \vec{e}_{k'} = k^2 \delta_{kk'}$

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This last expression gives the energy as a sum over the modes.

To quantize the field is better to write H in Hamiltonian form.

Let us introduce a pair of real canonical variables.
 $q_{ks}(t)$ and $p_{ks}(t)$

$$q_{ks}(t) = [u_{ks}(t) + u_{ks}^*(t)]$$

$$p_{ks}(t) = \dot{q}_{ks}(t) - i\omega [u_{ks}(t) - u_{ks}^*(t)]$$

q and p oscillate sinusoidally in time at frequency ω (remember $u_{ks}(t) = c_{ks} e^{-i\omega t}$)

$$\frac{\partial}{\partial t} q_{ks}(t) = p_{ks}(t)$$

$$\frac{\partial}{\partial t} p_{ks}(t) = -\omega^2 q_{ks}(t)$$

Then the energy can be written as.

$$H = \frac{1}{2} \sum_k \sum_s [p_{ks}^2(t) + \omega^2 q_{ks}^2(t)]$$

This is a system of independent harmonic oscillators. One for each k, s mode

The EM classical radiation field is specified by the set of all canonical variables $q_{k\sigma}(t)$, $p_{k\sigma}(t)$.
The set is countably infinite.

The Causal equations of motion are:

$$\frac{\partial H}{\partial p_{k\sigma}} = \frac{\partial q_{k\sigma}}{\partial t} \quad \frac{\partial H}{\partial q_{k\sigma}} = -\frac{\partial p_{k\sigma}}{\partial t}$$

Then we can express:

$$\hat{A}(\vec{r}, t) = \frac{1}{2\epsilon_0^{1/2} L^{3/2}} \sum_k \sum_s \left\{ [q_{k\sigma}(t) + i p_{k\sigma}(t)] \hat{E}_{k\sigma} e^{i\vec{k} \cdot \vec{r}} + \text{c.c.} \right\},$$

$$\hat{E}(\vec{r}, t) = \frac{i}{2\epsilon_0^{1/2} L^{3/2}} \sum_k \sum_s \left\{ [\omega q_{k\sigma}(t) + i p_{k\sigma}(t)] \hat{E}_{k\sigma} e^{i\vec{k} \cdot \vec{r}} - \text{c.c.} \right\}$$

$$\hat{B}(\vec{r}, t) = \frac{i}{2\epsilon_0^{1/2} L^{3/2}} \sum_k \sum_s \left\{ \left[q_{k\sigma}(t) + \frac{i}{\omega} p_{k\sigma}(t) \right] \vec{k} \times \hat{E}_{k\sigma} e^{i\vec{k} \cdot \vec{r}} - \text{c.c.} \right\}$$

Canonical quantization of the transverse field
Dirac, Heitler, Lorentz

$$q_{ks}(t), p_{ks}^{\pm}(t) \rightarrow \hat{q}_{ks}(t), \hat{p}_{ks}(t)$$

canonically conjugate operators.
with non zero commutator for $i \neq j$

As the classical variables associated with two different modes
are uncoupled, the corresponding Hilbert space operators
commute.

$$[\hat{q}_{ks}(t), \hat{p}_{k's'}(t)] = i\hbar S_{kk'}^3 \delta_{ss'}$$

$$[\hat{q}_{ks}(t), \hat{q}_{k's'}(t)] = 0$$

$$[\hat{p}_{ks}(t), \hat{p}_{k's'}(t)] = 0$$

The state is described by a $|k\sigma t/4\rangle$ in Hilbert space.

Now one can calculate expectation values of operators.

Physical observables are always associated with Hermitian
operators and the corresponding eigenvalues are all real.

Just consider the dynamical variables as Hilbert space
operators which do not all commute, all the equations
of motion remain valid as operator equations.

The Hamiltonian of the quantized radiation field is:

$$\hat{H} = \frac{1}{2} \sum_k \int \left[\hat{\rho}_{ks}^*(t) + \omega^2 \hat{q}_{ks}^2(t) \right]$$

Note that \vec{r} and t only play the role of parameters.

Let us define a set of non-Hermitian operators (for convenience).

$$\hat{a}_{ks}^*(t) = \frac{1}{(2\hbar\omega)^{1/2}} [\omega \hat{q}_{ks}(t) - i \hat{\rho}_{ks}(t)]$$

$$\hat{a}_{ks}^+(t) = \frac{1}{(2\hbar\omega)^{1/2}} [\omega \hat{q}_{ks}(t) + i \hat{\rho}_{ks}(t)]$$

inverting the equations

$$\hat{q}_{ks}(t) = \left(\frac{\hbar}{2\omega}\right)^{1/2} [\hat{a}_{ks}(t) + \hat{a}_{ks}^+(t)]$$

$$\hat{\rho}_{ks}(t) = i \left(\frac{\hbar\omega}{2}\right)^{1/2} [\hat{a}_{ks}^+(t) - \hat{a}_{ks}(t)]$$

$$[\hat{a}_{ks}(t), \hat{a}_{k's'}^+(t)] = \delta_{kk'}^3 \delta_{ss'}$$

$$[\hat{a}_{ks}(t), \hat{a}_{k's'}(t)] = 0$$

$$[\hat{a}_{ks}^+(t), \hat{a}_{k's'}^+(t)] = 0$$

The operators \hat{a}_{ks} and \hat{a}_{ks}^+ (t) except for the prefactor $(\frac{\hbar}{2\omega})^{1/2}$

correspond to the complex amplitudes

$$\hat{a}_{ks}(t) \text{ and } \hat{a}_{ks}^+(t) \quad u_{ks}(t) \text{ and } u_{ks}^*(t)$$

They have the very same time dependence.

$$\hat{a}_{ks}(t) = \hat{a}_{ks}(0) e^{-i\omega t}$$

$$\hat{a}_{ks}^+(t) = \hat{a}_{ks}^+(0) e^{i\omega t}$$

The product is time independent.

$$\text{Hence } \hat{H} = \frac{1}{2} \sum_{ks} \hbar \omega [\hat{a}_{ks}(t) \hat{a}_{ks}^+(t) + \hat{a}_{ks}^+(t) \hat{a}_{ks}(t)]$$

The operators appear in symmetrized form with respect to their order.

using $[\hat{a}, \hat{a}^+] = 1$ the commutation relation.

$$\hat{H} = \sum_{ks} \hbar \omega [\hat{a}_{ks}^+(t) \hat{a}_{ks}(t) + \frac{1}{2}]$$

This is normally ordered

$\hat{a}_{ks}^+(t)$ to the left of $\hat{a}_{ks}(t)$
anti-normal $[\hat{a}, \hat{a}^+ - \frac{1}{2}]$

This will be used thoroughly when talking about detection $\hat{e} \rightarrow$ annihilates.

The $\frac{1}{2} \hbar\omega$ is the zero point contribution.

From the uncertainty principle, a quantum-mechanical harmonic oscillator can never come to rest.

It has the unfortunate consequence for an unbounded set of modes of giving an infinite contribution to the energy. This is a difficulty of QED

This argument: in finite to very large wave numbers are unphysical so for all physically meaningful problems the sum $\sum_k \frac{1}{2} \hbar\omega$ should be finite

if k is sufficiently large $\hbar\omega$ may be larger than the energy of the universe.

Another approach is to abide ourselves by the correspondence principle.

This principle demands only that the results given by quantum theory agree with those of the classical theory in the classical limit, when the excitations become very large.

So $\frac{1}{2}$ may be neglected compared with $\hat{a}^\dagger \hat{a}$

so

$$\hat{H} = \sum_{kS} \hbar\omega \hat{a}_k^\dagger(t) \hat{a}_k(t)$$

Sometimes we shall only take this expression

Spectrum of the energy; photons.

$$\text{Classically } H = \sum_{\mathbf{k}s} \omega^2 / u_{\mathbf{k}s}(t) / 2$$

This expression or the one in terms of p and q admits all possible non-negative values for the energy.

The operator expression gives a different result.
October 4, 2007.

(Drop the time dependence understanding that all operators are evaluated at the same time).

$$\hat{a}_{\mathbf{k}s}^+ \hat{a}_{\mathbf{k}'s'} = \hat{n}_{\mathbf{k}s}$$

Let us find the spectrum of $\hat{n}_{\mathbf{k}s}$.

$$\begin{aligned} [\hat{a}_{\mathbf{k}s}, \hat{n}_{\mathbf{k}s}] &= \hat{a}_{\mathbf{k}s} \hat{a}_{\mathbf{k}s}^+ - \hat{a}_{\mathbf{k}s}^+ \hat{a}_{\mathbf{k}s} \hat{a}_{\mathbf{k}s} \\ &= [\hat{a}_{\mathbf{k}s}, \hat{a}_{\mathbf{k}s}^+] \hat{a}_{\mathbf{k}s} \\ &= \hat{a}_{\mathbf{k}s} \delta^3_{\mathbf{k}\mathbf{k}'} \delta_{ss'} \end{aligned}$$

$$\text{also } [\hat{a}_{\mathbf{k}s}^+, \hat{n}_{\mathbf{k}'s'}] = -\hat{a}_{\mathbf{k}'s'}^+ \delta^3_{\mathbf{k}\mathbf{k}'} \delta_{ss'}$$

Now we have to find the eigenvalues of $\hat{n}_{\mathbf{k}s}$.

The time dependence of $\hat{a}_{kz}(t)$ follow the Heisenberg equations of motion.

$$\frac{d\hat{O}}{dt} = \frac{1}{i\hbar} [\hat{O}, \hat{H}]$$

$$\frac{d\hat{a}_{kz}(t)}{dt} = -i\omega \hat{a}_{kz}(t)$$

$$\frac{d\hat{a}_{kz}^+(t)}{dt} = i\omega \hat{a}_{kz}^+(t)$$

with the solutions $\hat{a}_{kz} e^{-i\omega t}$ and $\hat{a}_{kz}^+ e^{i\omega t}$

so we can write the E.M. field operators

$$\hat{E}(\vec{r}, t) = \frac{1}{2\pi} \sum_k \sum_s \left(\frac{q\omega}{2\epsilon_0} \right)^{1/2} [i\hat{a}_{kz}(0) \epsilon_{ks} e^{i(k \cdot \vec{r} - \omega t)} + h.c.]$$

$$\hat{B}(\vec{r}, t) = \frac{1}{2\pi} \sum_k \sum_s \left(\frac{q}{2\omega \epsilon_0} \right)^{1/2} [i\hat{a}_{kz}^+(0) (\vec{k} \times \vec{\epsilon}_{ks}) e^{i(k \cdot \vec{r} - \omega t)} + h.c.]$$

Let us look for a moment at one mode, linear polarisation

$$\hat{E}(\vec{r}, t) = i\vec{\epsilon} \left(\frac{q\omega}{2\epsilon_0 L^3} \right)^{1/2} \{ \hat{a} e^{i(k \cdot \vec{r} - \omega t)} - \hat{a}^+ e^{-i(k \cdot \vec{r} - \omega t)} \}$$

Homework: Note the meaning of the constant $\vec{\epsilon}$.

from E84. The energy density is $\epsilon_0 E^2$

what would be the energy density of one photon.

$\frac{\hbar\omega}{V}$ to the field of one photon.

$$E = \left(\frac{\hbar\omega}{\epsilon_0 V} \right)^{1/2}$$

The constant is $\left(\frac{\hbar\omega}{\epsilon_0 V} \right)^{1/2}$

some people relate it to the vacuum. That is the expectation value of the $\frac{1}{2}$ ~~E~~ in the Hamiltonian.

Watch out in the literature. There are many names given to this constant. But it has a definite relationship with the field ~~E~~ amplitude of a single photon.

Again look at the $\langle |\vec{E}|^2 \rangle$ of a Fock state.

$$\langle n | |\vec{E}|^2 | n \rangle = \frac{\hbar\omega}{\epsilon_0 V} (n + \frac{1}{2}) \quad \text{independent of time! all the } \vec{e} \text{ are } e^{i\omega t} \text{ and!}$$

the fluctuations

$$\Delta E = \sqrt{\langle |\vec{E}|^2 \rangle - \langle \vec{E} \rangle^2} \Big|_{n=0}$$

$$= \left(\frac{\hbar\omega}{\epsilon_0 V} \right)^{1/2} (n + \frac{1}{2})^{1/2}$$

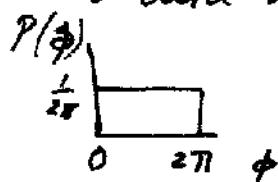
no cycle average is necessary

3 Is there a classical equivalent?

$$E = E_0 \cos(\omega t - \phi)$$

random phase.

The phase is random with equal probability between 0 and 2π



$$\begin{aligned}\bar{E}(t) &= \int P(\phi) E \, d\phi \\ &= E_0 \int_{2\pi}^{\frac{1}{2}} \cos(\omega t - \phi) \, d\phi = 0\end{aligned}$$

The result is independent of time.

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$

However,

the (rms)² of the field

$$\bar{E}^2(t) = E_0^2 \int_0^{2\pi} \frac{1}{2\pi} \cos^2(\omega t - \phi) \, d\phi$$

remember $\int_0^{2\pi} \cos^2 \phi \, d\phi = \pi$

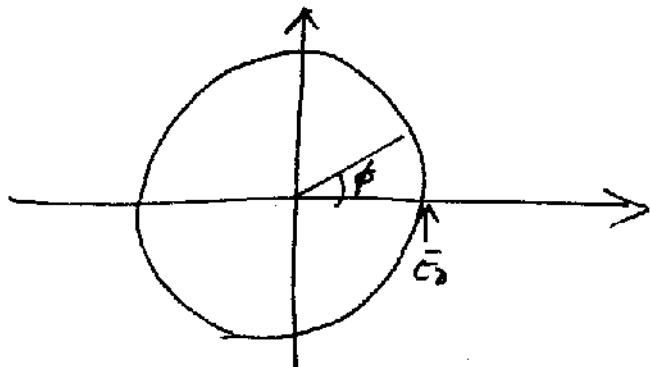
$$= \frac{E_0^2}{2}$$

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The deviations are $\Delta E = \frac{\epsilon_0}{\sqrt{2}}$

They also grow with \sqrt{energy} .

Fork state $\Delta E = \left(\frac{4\omega}{\epsilon_0} (n + 1/2) \right)^{1/2}$



Phase diagram:

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The Hilbert space for the EM field is :

$$\prod_{k,s} |n_{ks}\rangle = |n\rangle$$

where

$$|n\rangle = |n_1, n_2, \dots, n_s, \dots\rangle$$

$$\hat{n}_{ks} |n\rangle = n_{ks} |n\rangle \text{ eigenstate of } \hat{n}_{ks}$$

The total number operator

$$\hat{N} = \sum_{k,s} \hat{n}_{ks}$$

Then $\hat{N} |n\rangle = n |n\rangle$ eigenstate of \hat{N}
tot.

$|0\rangle$ = vacuum state $|\text{vac}\rangle$

Any Fock state can be generated by repeated application
of \hat{a}_{ks}^\dagger on $|\text{vac}\rangle$

$$|nm\rangle = \prod_{k,s} \frac{(\hat{a}_{ks}^\dagger)^{n_{ks}}}{\sqrt{(n_{ks})!}} |\text{vac}\rangle$$

$$\hat{H} / \langle n \rangle = \sum_{k,s} [\epsilon_{ks} \omega + \frac{\hbar^2}{2}] / \langle n \rangle$$

Note that the Fock states are eigenvectors of \hat{H} . These eigenvalues are discrete.

There are many eigenvectors with the same eigenvalue - the degeneracy as the number of modes increases becomes infinite.

The discrete excitations or quanta of the EM field corresponding to the occupation numbers $\{n\}$ are usually known as photons.

The eigenvalues of the photon number operator \hat{n}_{ks} are unbounded, an arbitrarily large number of photons may be found in the same quantum state, which means the photons are bosons and obey Bose-Einstein statistics.

The energy is independent of s and k , it only depends on ω .

$$\sum_{k,s} \frac{\epsilon_{ks} \omega}{\langle n \rangle}$$

an energy eigenvalue.

"Since a photon only interacts with itself" Dirac.

Plane waves are ideal for such experiments. This implies a well defined \vec{k} vector. &

The Fock states form a basis.

$$\underline{\alpha} = \sum_{\{n\}} | \{n\} \rangle \langle \{n\} |$$

They are orthonormal.

$$\langle \{n\} | \{m\} \rangle = \prod_{k,s} \delta_{n_k m_k}$$

Representation of

$$\hat{a}_{ks} = \sum_{\{n\}} (n_{ks})^{1/2} | n_{ks} - 1 \rangle \langle n_{ks} | \prod_{k' \neq k, s'} | n_{k's'} \rangle \langle n_{k's'} |$$

and

$$\hat{a}_{ks}^+ = \sum_{\{n\}} (n_{ks} + 1)^{1/2} | n_{ks} + 1 \rangle \langle n_{ks}' | \prod_{k' \neq k, s'} | n_{k's'} \rangle \langle n_{k's'} |$$

entirely off-diagonal k', s'

The expectation value for a Fock state!

$$\langle \{n\} | \hat{a}_{ks} | \{n\} \rangle = 0 = \langle \{n\} | \hat{a}_{ks}^+ | \{n\} \rangle$$

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It is difficult to say ~~that~~ that a Fock state is an easily solvable field.

We would like to create a very classical state.

- something we could relate to.
- measuring does not disturb it
- we know that detection is traditionally by annihilation of photons

(Introduced by R. J. Glauber)

Phys. Rev. A 130, 2529 (1963)

131, 2266 (1963)

already E. Schrödinger talked about them as minimum uncertainty states.
at least there.

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

where α is a complex number.

The operation of annihilating a photon will not affect the state.

Try to construct them using Fock states.

$$|\alpha\rangle = \sum_n C_n |n\rangle$$

^{changed}
_{index}

$$\hat{a}|\alpha\rangle = \hat{a} \sum_{n=0}^{\infty} C_n |n\rangle = \sum_{n=1}^{\infty} C_n \sqrt{n} |n-1\rangle = \alpha \sum_{n=0}^{\infty} C_n |n\rangle$$

Start with $\langle m \rangle$ to calculate the value of the coefficients C_n

$$\alpha \sum_{n=0}^{\infty} C_n \delta \langle m | n \rangle = \sum_{n=1}^{\infty} C_n \delta \langle m | n-1 \rangle$$

$$C_0 \sqrt{n} = \alpha C_{n-1}$$

$$C_n = \frac{\alpha C_{n-1}}{\sqrt{n}}$$

$$C_{n+1} = \frac{\alpha C_n}{\sqrt{n+1}} \dots \dots \quad C_n = \frac{\alpha^n C_0}{\sqrt{n!}}$$

$$|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$\langle \alpha | = C_0 \sum_{m=0}^{\infty} \frac{\alpha^{*m}}{\sqrt{m!}} \langle m |$$

Let us now find the normalization constant

$$\langle \alpha | \alpha \rangle = 1 = |C_0|^2 \sum_{m,n} \frac{\alpha^{*m} \alpha^n}{\sqrt{n! m!}} \langle m | n \rangle$$

↑
Delta

$$\langle \alpha | \alpha \rangle = |C_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^n}{n!} \quad \leftarrow \text{exponential expansion}$$

$$|C_0|^2 = e^{-|\alpha|^2}$$

$$C_0 = e^{-|\alpha|^2/2}$$

with some arbitrary phase left undetermined.

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

In order to construct a coherent state all words are infinite number of field states "phased" in a very special way as to produce a "classical looking state" that is not expected by measuring one photon.

What is the probability that a measurement of the energy can yield $E_n = (n + 1/2) \hbar \omega$ if the field is in a coherent state $|\alpha\rangle$?

$$P_n(\alpha) = |\langle n | \alpha \rangle|^2 \quad \langle n | \alpha \rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$$

$$= |\langle n | e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} | \alpha \rangle|^2$$

numbers

so

$$\langle n^2 \rangle - \langle n \rangle^2 = \beta^2 + \beta - \beta^2 = \beta.$$

$$\langle n^2 \rangle - \langle n \rangle^2 = |\alpha|^2$$

The Variance is equal to the mean
the standard deviation is $|\alpha|$.

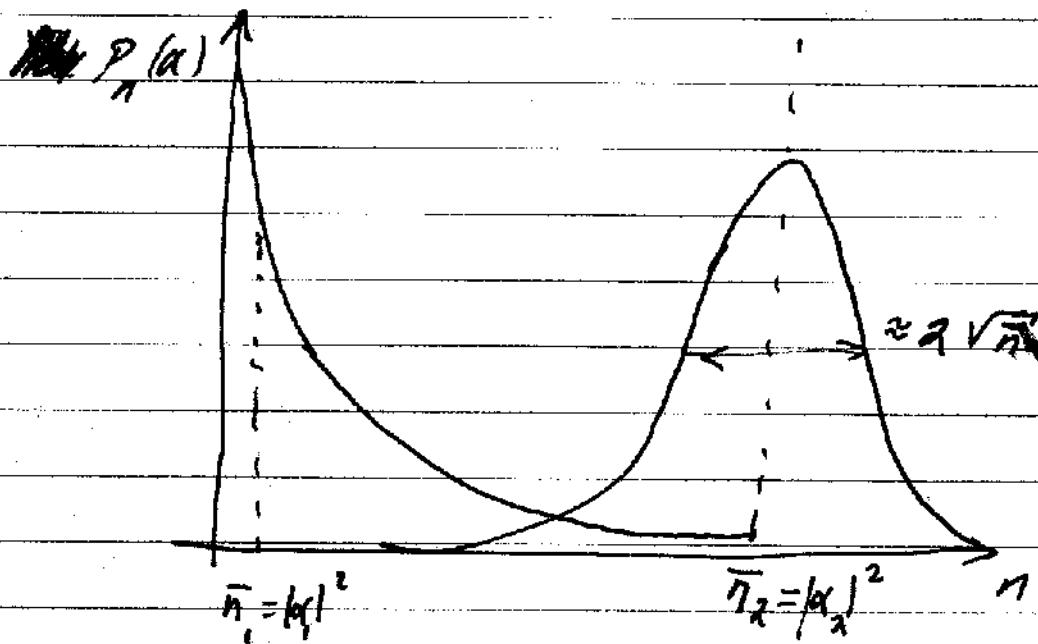
The noise is a coherent state.

$$\frac{\text{St. Dev.}}{\text{Mean}} = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$$



Poisson distribution.

This is the "shot noise" very classical.



approaches a Gaussian for large n .