

September 11, 2003

1.- The density operator for a TLA.

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

but we have only two states $|g\rangle, |e\rangle$.

so the density matrix is 2×2 .

$$\hat{\rho}_{ij} = \begin{pmatrix} |g\rangle\langle g| & |g\rangle\langle e| \\ |e\rangle\langle g| & |e\rangle\langle e| \end{pmatrix}$$

The diagonal elements measure the population n_{state} of the state $\psi = a|g\rangle + b|e\rangle$

$\rho_{22} = \rho_{11}$ ← probability in $|g\rangle$

probability of finding in measurement this system in $|e\rangle$ and the off diagonal elements. The coherences.

$$|g\rangle\langle g|g\rangle = |g\rangle$$

$$|e\rangle\langle e|g\rangle = 0$$

$$|g\rangle\langle e|g\rangle = 0$$

$$|e\rangle\langle g|g\rangle = |e\rangle$$

↑
interference effects between
 $|g\rangle$ and $|e\rangle$

$$\rho_{11} \Rightarrow ab^*$$

$$\rho_{22} \Rightarrow a^*b$$

This is just another formulation that can be mapped to the spin problem but you can read more readily what you are doing.

This is also a very convenient way to work since it will be the way to introduce formally dissipation. Remember some of its properties for a pure state.

$$\text{Tr } \rho(t) = 1$$

$$\langle A \rangle(t) = \text{Tr} \{ \rho(t) A \}$$

$$\frac{d}{dt} \rho(t) = \frac{i}{\hbar} [H(t), \rho(t)]$$

$$\rho^+(t) = \rho(t) \quad \rho^2(t) = \rho(t) \quad \text{Tr } \rho^2(t) = 1$$

There are other operator representations that are very useful.

creation and annihilation of atomic excitation.

\hat{b}^+ creates an atom in the ground state

\hat{b} annihilates an atom in the ground state

\hat{c}^+ creates an atom in the excited state

\hat{c} ~~creates~~ annihilates an atom in the ground state

Inversion $\frac{\hat{c}^+ \hat{c}}{\pi} - \frac{\hat{b}^+ \hat{b}}{\pi}$

atoms in excited atoms in ground.

$$P_{ee} - P_{ii}$$

The Rabi Problem:

The differential equations are:

$$\ddot{u} = -\Delta u$$

$$\ddot{v} = +\Delta u + k \epsilon w$$

$$\ddot{w} = -k \epsilon v$$

Now on resonance when $\Delta = 0$ we have for $\epsilon = \epsilon(t)$

$$\ddot{u} = 0 \Rightarrow u = u_0$$

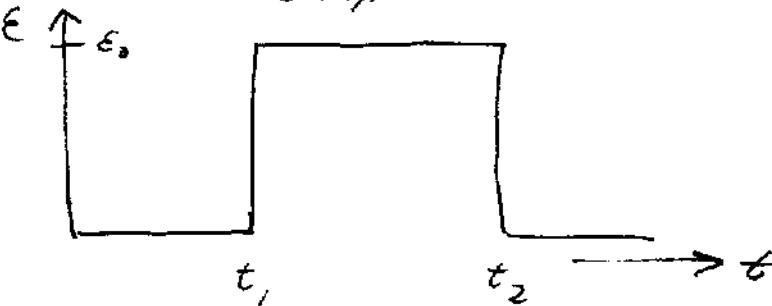
$$\ddot{v} = k \epsilon w \Rightarrow v(t, 0) = V_0 \sin \theta(t) + V_0 \cos \theta(t)$$

$$\ddot{w} = -k \epsilon v \quad w(t, 0) = -V_0 \sin \theta(t) + W_0 \cos \theta(t)$$

where $\theta(t) = \int_{-\infty}^t k \epsilon(t') dt'$

This is the Rabi solution.

Take the simple case:



$$\theta(t) = k \epsilon_0 (t_2 - t_1)$$

Frequency:

$$\frac{2 \pi \epsilon_0}{\hbar} = \Omega_0$$

Rabi frequency.

Please note that it combines a FIELD and a dipole moment for an atom.

we can think of this problem in terms of the Bloch vector

$$\vec{\rho} = (u, v, w)$$

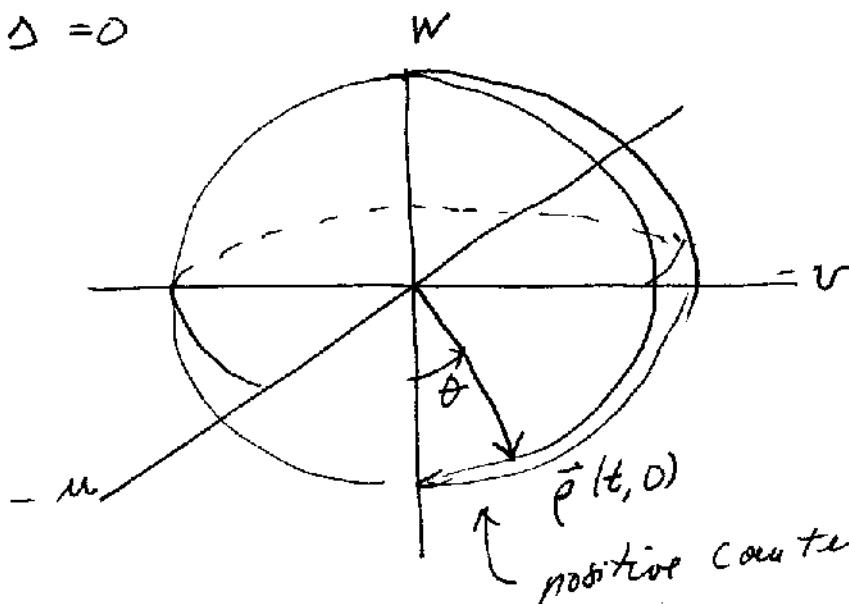
$\xrightarrow{\text{inversion}}$
 $\xleftarrow{\text{coherenc}} \circ$

Now the differential equations are these.

$$\frac{d\vec{\rho}}{dt} = \vec{\Omega} \times \vec{\rho} \quad \vec{\Omega} = (-k\epsilon, 0, \Delta)$$

$$\frac{d}{dt} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & k\epsilon \\ 0 & -k\epsilon & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

for $\Delta = 0$



note some things
very important
if you get it
at $\underline{\underline{\pi/2}}$.

The Torque is in the u axis. The ~~Bloch~~ Vector just goes up and down around u

so if $\theta(t) = \frac{\pi}{2}$ the state goes from $|g\rangle$ to $|e\rangle$
 if it is $\theta(t) = \pi$ then you have a rotation all
 around to return to the ground state

A very interesting angle is $\pi/2$ you leave the atom
 in a superposition of $|g\rangle + |e\rangle$ with a lot of
 coherence.

Note: this manipulation is key to Quantum Computing, but
 please note that any dissipation would destroy the
 super-coherent sum \rightarrow keep the phase.

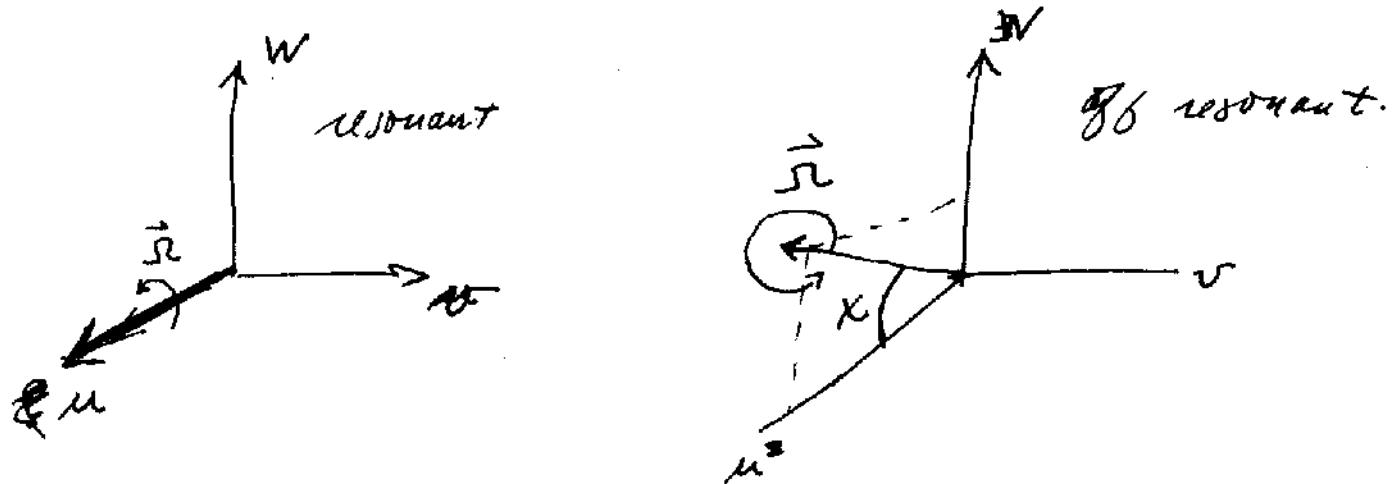
The solution is a rotation of angle $\theta(t)$
 around $-u$

let us continue with the Rabi problem $\underline{E_0}$ with
 detunings:

It is possible to reduce this new problem

$$\frac{d}{dt} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 & -\Delta & 0 \\ \Delta & 0 & \kappa c \\ 0 & -\kappa c & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

into one of the form we have already seen by a rotation of \vec{q} around the 2nd (v) axis.



$$\tan X = \frac{\Delta}{kE_0}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos X & 0 & \sin X \\ 0 & 1 & 0 \\ -\sin X & 0 & \cos X \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

now

$$\frac{d}{dt} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega(\Delta) \\ 0 & -\Omega(\Delta) & 0 \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

$$\text{where } \Omega(\Delta) = \sqrt{\Delta^2 + (kE_0)^2}$$

which is the problem we already know how to solve. \rightarrow so the final solution.

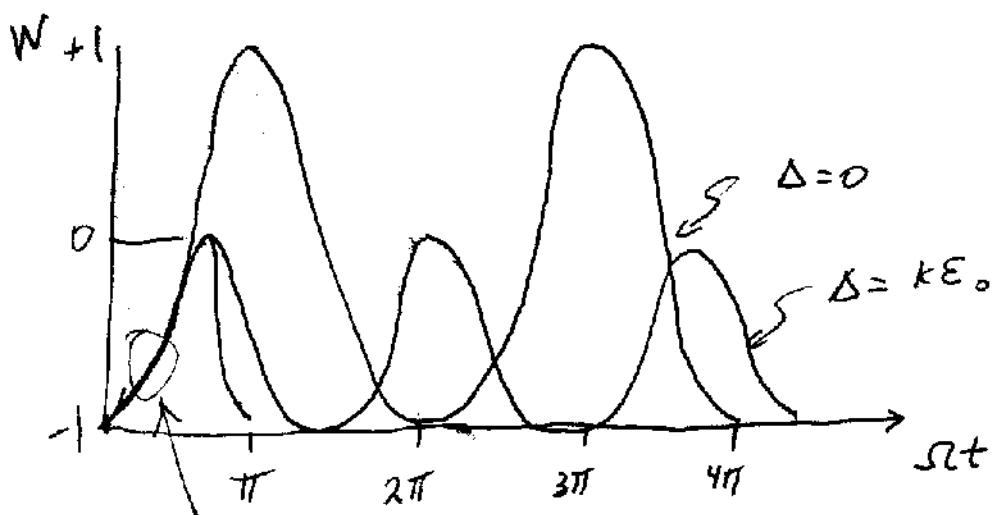
$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos X & 0 & \sin X \\ 0 & 1 & 0 \\ -\sin X & 0 & \cos X \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Omega t - \sin \Omega t & \sin \Omega t \\ 0 & \sin \Omega t & \cos \Omega t \end{bmatrix} \begin{bmatrix} \cos \Delta & 0 & -\sin \Delta \\ 0 & 1 & 0 \\ \sin \Delta & 0 & \cos \Delta \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \frac{(k\epsilon_0)^2 + \Delta^2 \cos \Omega t}{\Omega^2} & -\frac{\Delta}{\Omega} \sin \Omega t & -\frac{\Delta k\epsilon_0}{\Omega^2} (1 - \cos \Omega t) \\ \frac{\Delta}{\Omega} \sin \Omega t & \cos \Omega t & \frac{k\epsilon_0}{\Omega} \sin \Omega t \\ -\frac{\Delta k\epsilon_0}{\Omega^2} (1 - \cos \Omega t) & -\frac{\Delta k\epsilon_0}{\Omega} \sin \Omega t & \frac{\Delta^2 + (k\epsilon_0)^2 \cos \Omega t}{\Omega^2} \end{bmatrix} \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix}$$

The length is measured in time, all we have are rotations.
 if the atom is initially in the ground state then $w_0 = -1$
 $u_0 = v_0 = 0$

then

$$w = -\frac{\Delta^2 + (k\epsilon_0)^2 \cos \Omega t}{\Omega^2} = -\frac{\Delta^2 + (k\epsilon_0)^2 (1 - 2 \sin^2 \frac{\Omega t}{2})}{\Omega^2} \\ = -1 + \frac{2(k\epsilon_0)^2}{(k\epsilon_0)^2 + \Delta^2} \sin^2 \sqrt{(k\epsilon_0)^2 + \Delta^2} \frac{t}{2}$$



note they start equally but quickly separate

Remember that we have been talking about the solutions in the rotating frame. What happens to \vec{s} is different. (Take the resonant case).

\vec{s} precesses very rapidly at frequency ω about the 3 axis and we have to superimpose the up and down from -1 to +1 in w (mutation)

Now we have to talk about damping. We know atoms decay back to the ground state. (some typical T_1 are $\approx 20 \mu\text{s}^{30}$ for the D_2 lines of alkali)

We have to include it in the Bloch equations

$$\ddot{u} = -\Delta v - \frac{u}{T_2} \quad \text{or} \quad \ddot{u} = -\delta v - \gamma_1 u$$

$$\dot{v} = \Delta u - v + k E w$$

$$\dot{v} = \delta u - \frac{\gamma_1}{k} \delta_1 v + k E w$$

$$\dot{w} = -\frac{w - w_{eq}}{T_1} - k E v$$

$$\dot{w} = -\delta_{11}(w - w_{eq}) - k E v$$

longitudinal.

The solutions are still possible in the Rabi case. (constant amplitude). They will be sines and cosines attenuated

Now it is important to find the steady state solution.

$$x = v = w = 0$$

$$W_{ss} = W_{eq} \frac{1 + (\Delta T_2)^2}{1 + (\Delta T_2)^2 + T_1 T_2 (k \epsilon_0)^2}$$

$$V_{ss} = W_{eq} \frac{k \epsilon_0 T_2}{1 + (\Delta T_2)^2 + T_1 T_2 (k \epsilon_0)^2}$$

$$\bullet u_{ss} = -W_{eq} \frac{(\Delta T_2)(k \epsilon_0 T_2)}{1 + (\Delta T_2)^2 + T_1 T_2 (k \epsilon_0)^2}$$

The general solution is complicated but it has the form:

$$A e^{-at} + B \left[\cos \tilde{\omega} t + \frac{C}{\tilde{\omega}} \sin \tilde{\omega} t \right] e^{-bt} + D$$

where now $\tilde{\omega}$ is some value related to the ^{ss.} Resonance frequency.

an important case: Resonance.

$$a = \frac{1}{T_2} \quad b = \frac{1}{2} \left(\frac{1}{T_1} + \frac{1}{T_2} \right) = \frac{1}{2} (\gamma_1 + \gamma_2)$$

$$\tilde{\omega} = \sqrt{(k \epsilon_0)^2 - \frac{1}{4} (\gamma_1 - \gamma_2)^2}$$

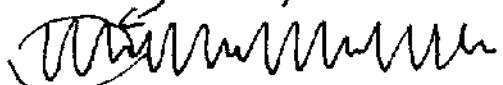
\uparrow average diff freq.

"impedance mismatch"

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Now we do not keep going around and around \oplus
 after some time $\approx \frac{T_1 + T_2}{2}$ the atom decays and
 stops its Rabi oscillations.

What is happening to the atom: we are turning on
 and off the dipole. (ground state and excited state
 have no dipole)

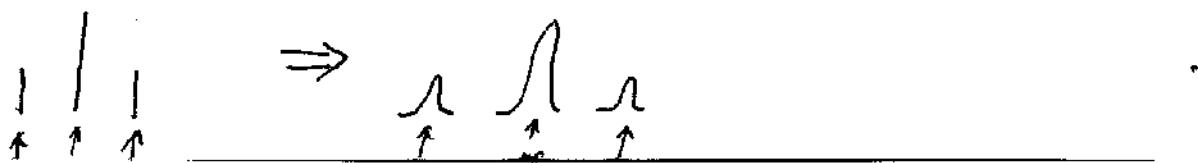
 amplitude modulation of the dipole.

How does the spectrum look
 for an AM signal?

$A \sin \omega t \rightarrow$ ~~$A_0 \sin(\omega t + \phi)$~~ $\sin \omega t \Rightarrow$ has components
 $A_0 (1 + \eta \sin \omega t) \sin \omega t$
 at $\underline{\omega}, \underline{\omega + \Delta \omega} \uparrow, \underline{\omega - \Delta \omega} \uparrow$
 AM modulation.

How would you measure that \rightarrow "Radio"
 Look at the homodyne signal between the dipole and
 the drive.

The presence of decay now puts a bandwidth $\Delta \omega$.



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The solutions we have looked at are not the most general to the problem.

An interesting case is to make the parameter Δ vary in time \rightarrow Then one can do some adiabatic following.

Please remember that the evolution of the Bloch vector \vec{s} on the Bloch sphere is quite complicated, it may or may not describe a simple pattern. Accumulate non-trivial phases!

Let us remember what form \vec{E} takes. The Envelope of the electric field follows Beer's law

$$\frac{\partial E}{\partial z} = -\frac{1}{2} \alpha E$$

\uparrow absorption coefficient associated with the properties of the atoms (dipoles)

~~•~~ $\alpha = \alpha(\omega)$ but it may happen that

$$\alpha = \alpha(\omega, |\vec{E}|^2)$$

\uparrow Intensity. This is saturation.

Although we will talk some more about

saturation. This idea is intimately related to
a comparison of the coherent rate to the
incoherent rate

↑
Rabi frequency

decay δ , through spontaneous emission.

$$\begin{aligned} I_{\text{sat}} &= \frac{\text{one photon energy}}{\text{cross section of the atom} \cdot \text{lifetime (spontaneous emission)}} \\ &= \frac{\pi \omega}{2\pi \lambda^2 \cdot 2\tau} \quad \approx \text{a few mW/cm}^2 \text{ for D}_2 \text{ line} \\ &\quad \text{of alkali.} \end{aligned}$$

subject to definition.