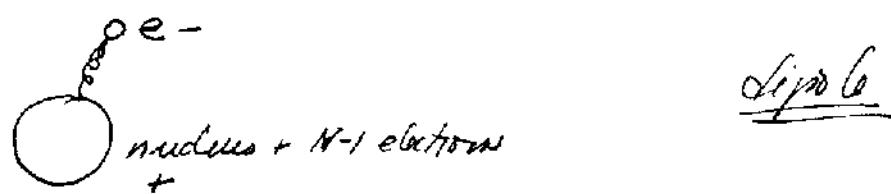


Sept. 4, 2003

Chapter 1 of Allen & Eberly.

Review of resonance theory of optics.

Lorentz Model.



Take it as an oscillator. Driven by the EM field. You can clearly see that the polarization does matter.

We are going to assume non-relativistic energies and velocities as well as low viscous constants (Faraday effect) so that only the electric field matters, the magnetic part does not. The term  $\frac{e}{c} \vec{v} \times \vec{B}$  is negligible.

array of dipoles of mass  $m$

$$H = \frac{1}{2m} \sum_i (\vec{p}_i^2 + \omega_i^2 m^2 \vec{r}_i^2) - e \sum_i \vec{r}_i \cdot \vec{E}(t, \vec{r}_i)$$

↑  
polarization response.

Hamilton's equations give us.

$$\ddot{x}_i + \omega_i^2 x_i = \frac{e}{m} \vec{E}(t, \vec{r})$$

↑ already scalar

This is just the driven H.O. that you know and have solved many times.

Now the problem is not yet consistent.

The dipole is driven  $\rightarrow$  radiates energy

The energy radiated into the field must be consistent with the energy lost by the oscillator. We are talking about radiation reaction.

$\rightarrow$  Remember conservation laws of EM (Griffiths, Jackson).

Sources of  
radiating vector

$$\nabla \cdot \vec{S} + \frac{\partial U_{em}}{\partial t} + \frac{\partial U_{matter}}{\partial t} = 0$$

electromagnetic matter  
energy density energy density.

So integrating on a small sphere around the oscillator  $\Rightarrow$

$$\int_A \vec{S} \cdot \hat{n} dA + \frac{\partial W_{em}}{\partial t} + \frac{\partial W_{mat}}{\partial t} = 0$$

Assume a very high  $Q$  for this oscillator.

$$Q = \frac{\omega_i \text{ frequency}}{\Delta \omega_i \text{ index of damping}} \gg 1$$

$\approx$  damping rate

we will assume that  $\frac{\partial W_{em}}{\partial t} \ll 1$  very small.

and since we have H.O.  $\Rightarrow W_i = m \omega_i^2 X_i^2(t)$  the mechanical energy

then:

$$\int \vec{S} \cdot \vec{n} dA = - \frac{\partial W_{\text{mech}}}{\partial t}$$

~~for + oscillations~~

From E&M we know. The energy loss by electric dipole radiation

$$\int \vec{S} \cdot \vec{n} dA = \frac{2e^2 w_i^4}{3c^3} \overline{X_i^2(t)}$$

see notes on Griffiths.

or written in another way:

$$\int \vec{S} \cdot \vec{n} dA = \frac{2e^2 w_i^2}{3c^3 m} W_i(t)$$

so that

$$\frac{\partial W_i(t)}{\partial t} = - \frac{2e^2 w_i^2}{3c^3 m} W_i(t) = - \frac{2}{\tau_0} W_i$$

This is a very simple equation.

$$W_i(t) = W_i(0) e^{-2t/\tau_0}$$

where  $\frac{1}{\tau_0} = \frac{e^2 w_i^2}{3mc^3}$   $\approx 100 \text{ n-1}$  for optical frequencies.

For a simple dipole then

$$\ddot{x}_i + \frac{2}{\tau_0} \dot{x}_i + w_i^2 x_i = \frac{e}{m} E$$

field actions on

The dipole because of all other charges and curr.

Now we can proceed to solve this problem in your favorite way.  $\rightarrow$  take the Fourier transform, look around resonance.

We are going to take a common wisdom path. Heat will pay back in making clearer the connection with Q.M.

Let  $E = E [e^{i\omega t} + e^{-i\omega t}] = 2E \cos \omega t$   
 $\uparrow$   
 constant.

$$x_i = x_0 [u_i \cos \omega t - v_i \sin \omega t]$$

$\uparrow$  in phas.       $\uparrow$   
 quadrature of the response

$u_i$  and  $v_i$  may vary very slowly in time, around 0.  
 all the fast oscillations are in  $\cos \omega t$ ,  $e^{i\omega t}$ , and  $\sin \omega t$ .

so:

$$\begin{array}{ll} \ddot{u}_i \ll \omega_i^2 u_i & \ddot{v}_i \ll \omega_i^2 v_i \\ \dot{v}_i \ll \omega_i v_i & \dot{u}_i \ll \omega_i^2 u_i \end{array}$$

Then the linear H.O. equation separates into two:  
 the cosine (in phase) and the sine (out of phase).  
 neglect  $\dot{u}_i$  and  $\dot{v}_i$

$$-x_0 [u_i \omega_i^2 + 2 \dot{v}_i \omega] + \frac{2}{\rho} x_0 [\dot{u}_i - v_i \omega] + \omega_i^2 x_0 u_i = \frac{2e}{m} E$$

$$-x_0 [-v_i \omega^2 + 2 \dot{u}_i \omega] - \frac{2}{\rho} x_0 [\dot{v}_i + u_i \omega] - \omega_i^2 x_0 v_i = 0$$

Then the sin part gives:

$$\ddot{u}_i = - \frac{(\omega_i^2 - \omega^2)}{2\omega} v_i - \frac{u_i}{\tau} - \frac{\dot{v}_i}{\omega \tau}$$

$\uparrow$   
neglect these terms

The cosine part gives:

$$\ddot{v}_i = \frac{u_i (\omega_i^2 - \omega^2)}{2\omega} - \frac{v_i}{\tau} - \frac{e E}{m \omega \chi_0} + \frac{\dot{u}_i}{\omega \tau}$$

Take the limit of very close to resonance:

$$\omega_i \approx \omega \quad \text{so} \quad \omega_i^2 - \omega^2 \approx 2\omega(\omega_i - \omega)$$

$$\text{and } \Delta_i \equiv \omega_i - \omega \quad \text{now drop the index.}$$

$$\ddot{u} = -\Delta v - \frac{u}{\tau} \quad \text{where } \tau \text{ is a generalized decay (more in a moment).}$$

$$\ddot{v} = -\Delta u - \frac{v}{\tau} - K E$$

Then the solutions are

$$u(t, \Delta) = [u_0 \cos \Delta t - v_0 \sin \Delta t] e^{-t/\tau} + K E \int_0^t dt' \sin \Delta(t-t') e^{-\frac{t-t'}{\tau}}$$

$$v(t, \Delta) = [u_0 \sin \Delta t + v_0 \cos \Delta t] e^{-t/\tau} + K E \int_0^t dt' \cos \Delta(t-t') e^{-\frac{(t-t')}{\tau}}$$

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From this we need to find the S.S solution for the dipole displacement  $x_i$

$$x_i(t) = \frac{e}{m} E \left( \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\frac{\omega}{T}} + cc \right)$$

The driven dipole oscillates at the driven frequency but not exactly in phase

Note that if  $\omega \gg \omega_i$  or if  $\omega \ll \omega_i$  there is a change of  $\pi$  in the phase. something you know from any driven H.O.

Now the emission is not necessarily from a single atom.  $\rightarrow$  There are many that oscillate and give rise to a Polarization density  $P(t)$ .

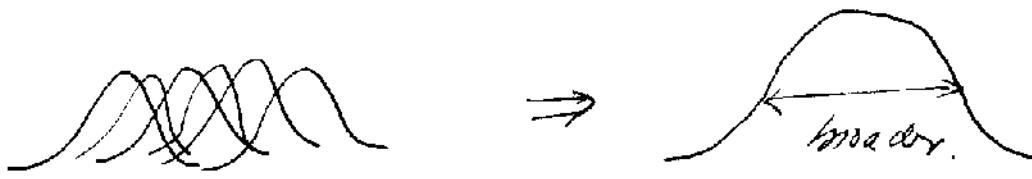
$\rightarrow$  Also note that this has a specific timeshape that is Lorentzian.

(say if you look at the energy). remember  $\omega_i \approx \omega$

$$\frac{1}{2\omega\Delta + 2i\frac{\omega}{T}} + cc = \frac{2\omega\Delta - 2i\frac{\omega}{T}}{(2\omega\Delta)^2 + \left(\frac{2\omega}{T}\right)^2} + cc$$

- Why should an oscillator have a different frequency from another if they are the same atom? The environment Broadening!  $\rightarrow$

Try to look at the energy emitted  $\rightarrow$  intensity emitted by many atoms with different  $\omega_0$  varies.



This is the origin of inhomogeneous broadening: each oscillator has a different  $\omega_i$  for example:

There is a magnetic field gradient  $\rightarrow$  the frequency depends on the position (this is the origin in NMR). Another very important mechanism is Doppler broadening.

$$\begin{array}{c} \uparrow \vec{\omega}_0 \\ \downarrow \oplus \end{array} \quad \xleftarrow[\text{propagation.}]{\vec{k} = \frac{2\pi}{\lambda} \hat{\vec{k}}} \text{EM. wave}$$

$$\text{First order Doppler} \rightarrow \omega \rightarrow \omega_0 + \vec{k} \cdot \vec{v}$$

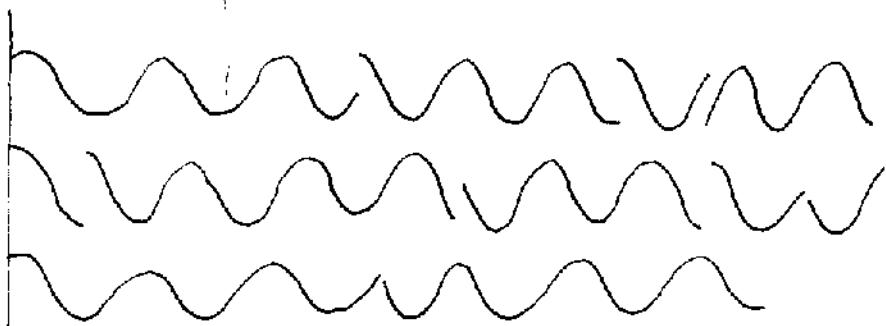
Each atom in a MB distribution has a different velocity so it is broadened.

- Since each dipole has different frequencies  $\rightarrow$  the waves can not add coherently for a long time they dissipate, but notice that each dipole is

still oscillating at its own frequency.

The dephasing of the dipoles could be "homogeneous" if they dephase the emission (collectively) is modulated.

Take collisions between atoms.

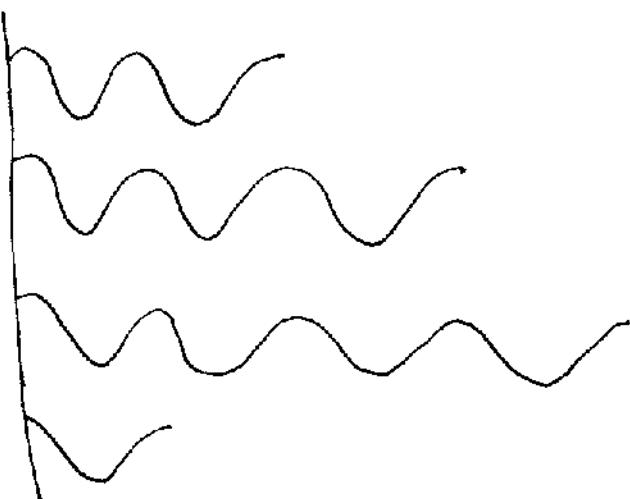



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only  
one is the  
result  
constructive

Homogeneous broadening  $\rightarrow$  all atoms affected the same  $\rightarrow$  dephasings of the dipoles. Again the dipoles are still oscillating, only the phase is randomized and is gone.

The natural decay dephasings (spontaneous emission) is a homogeneous process.



In general we have to introduce a distribution of frequencies  $\rightarrow$  inhomogeneous lineshape function (normalized)

$G(\omega_0)$  where  $G(\omega_0)d\omega_0$  is the fraction of dipole with resonance at the frequency  $\omega_0 \Rightarrow \omega_0 + d\omega_0$  such that  $\int_0^\infty G(\omega_0')d\omega_0' = 1$

we can shift to a detuning function  $g(\Delta)$

$$\Delta = \omega_0 - \omega$$

$$\int_{-\infty}^{\infty} g(\Delta')d\Delta' = 1$$

& extended assuming a peaked function.

In general there will be polarization density for a group of dipoles with density  $n$

$$P(t) = n \cdot \chi_0 \int \text{Re} \left[ [u(t, \Delta') + i v(t, \Delta')] e^{i\omega t} \right] g(\Delta') d\Delta'$$

if there is no ~~modulation~~ modulation, then  $g(\Delta') \rightarrow \delta(\Delta')$

and we recover the old result, otherwise we have

$\rightarrow$  see 9a for the solution

to perform this integral  $\rightarrow$  convolution in F. T talkies about spectrum.

In general if  $g$  is Lorentzian with width  $\Gamma$  (which are Lorentzian)  
and the dipoles have a width  $\delta$  the result is

$$P_{\text{TOT}} = P + \delta$$

9a

if on the other hand  $g(\Delta')$  is very broad. with  
a Lorentzian of width  $\delta\omega$

$$P(t) = \eta e x_0 e^{-t/\tau} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{1}{(\Delta' - \Delta)^2 + (\delta\omega)^2} \text{Re} \left[ (\mu_0 + i\nu_0) e^{i(\Delta't + \omega t)} \right] d\omega$$

$$= \eta e x_0 \text{Re} \left\{ (\mu_0 + i\nu_0) e^{-\Delta t} e^{i(\omega + \nu_0 t)} e^{-t/\tau} \frac{e^{-\delta\omega t}}{\delta\omega} \right\}$$

$\uparrow$  ordinary decay       $\uparrow$  fast decay.

Total Decay Time  $\frac{1}{\tau} + \frac{\delta\omega}{\delta\omega + \tau}$

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- if the two functions happen to be Gaussians  $\rightarrow$  Here.  
 $P_{tot} = \sqrt{\delta\omega^2 + \tau^2}$

if not  $\rightarrow$  Voigt Profile (Plasma Function)