Physics 711, Symmetry Problems in Physics Fall 2005

Homework solutions for assignment 5

Solutions for assignment due 11/22/05

1. The three group laws are

$$T(a_2)T(a_1) = T(a_2 + a_1), \tag{1}$$

$$L(\Lambda_2)L(\Lambda_1) = T(\Lambda_2\Lambda_1), \tag{2}$$

$$L(\Lambda)T(a) = T(\Lambda a)L(\Lambda), \tag{3}$$

or

$$L(\Lambda)T(\Lambda^{-1}a) = T(a)L(\Lambda), \tag{4}$$

Translations are abelian, so

$$T(a)|(p, \zeta\rangle = exp(ip \cdot a)|p, \zeta\rangle \tag{5}$$

From (4)

$$T(a)L(\Lambda)|p, \zeta\rangle = exp(ia \cdot \Lambda p)|(p, \zeta\rangle \tag{6}$$

so $\Lambda|p, \zeta\rangle$ carries energy-momentum Λp. Factor $\Lambda = \Lambda_3\Lambda_2\Lambda_1$, where $\Lambda^\dagger_{1,3} = \Lambda_{1,3}$, i.e., $\Lambda_{1,3}$ are boosts and $\Lambda^\dagger_2 = \Lambda^{-1}_2$, i.e., Λ_2 is a rotation. Pick a pure Lorentz transformation (a boost) Λ_1 that takes $|p, \zeta\rangle \to |p_0, \zeta\rangle$.

$$L(\Lambda_1)|p, \zeta\rangle = |p_0, \zeta\rangle \tag{7}$$

The Lorentz transformation that does this is (for the case of most interest, $p^2 = m^2,\ p^0 > 0$ so $p_0 = (m, \mathbf{p})$).

$$\Lambda_1 = \begin{pmatrix} cosh\lambda_1 & sinh\lambda_1 \\ sinh\lambda_1 & cosh\lambda_1 \end{pmatrix} \tag{8}$$

where $cosh\lambda_1 = E/m,\ \ sinh\lambda_1 = -|\mathbf{p}|/m$ and the boost is in the opposite direction to $\mathbf{p}$.

Now we rotate the state of the particle in its rest frame, (call $\Lambda_2 \equiv R$, since it is a rotation),

$$L(R)|p_0, \zeta\rangle = \sum_{\zeta'} |p_0, \zeta'\rangle D^{(j)}(R)_{\zeta',\zeta} \tag{9}$$

We don't yet know what R is; we will find that out at the end.
Next we Lorentz transform p_0 to p without changing ζ'.

$$L(\Lambda_3)|p_0, \zeta'\rangle = |\Lambda_3 p_0, \zeta'\rangle, \tag{10}$$

where $\Lambda_3 p_0 = \Lambda p$. Let $\mathbf{p}$ be in direction $\mathbf{n}'$. Then the boost from p_0 to $\Lambda_3 p$ is

$$\Lambda_3 = \begin{pmatrix} cosh\lambda_3 & sinh\lambda_3 \\ sinh\lambda_3 & cosh\lambda_3 \end{pmatrix} \tag{11}$$

where this matrix refers to the 0 and $\mathbf{n}'$ directions. Now we can find $R = \Lambda_3^{-1}\Lambda\Lambda_1^{-1}$.

2. Take the Dirac equation to be

$$[\gamma^\mu(i\partial_\mu - eA_\mu) - m\mathbf{1}]\Psi = 0 \tag{12}$$

Define

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{13}$$

Then

$$\Psi_L = \frac{1}{2}(1 - \gamma_5)\Psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \tag{14}$$

$$\Psi_R = \frac{1}{2}(1 + \gamma_5)\Psi = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{15}$$

Let $P_L = \frac{1}{2}(1 - \gamma_5)$, $P_R = \frac{1}{2}(1 + \gamma_5)$ be the chiral projection operators. Thus the Dirac spinor in the Weyl basis is

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix} \tag{16}$$

3. To make this discussion clear I will introduce a different notation than I used in class. Recall that we defined $X = \sigma^\mu x_\mu$. Since $X' = AXA^\dagger$, the sigmas must have one undotted and one dotted index, $(\sigma^\mu)_{\alpha\dot{\beta}}$. Then we cannot form the product $\sigma^\mu\sigma^\nu$! We can't sum an undotted and a dotted index together! If you ignore that you don't get a covariant expression for the product of two sigmas. You must introduce two types of sigmas, $(\sigma^\mu)_{\alpha\dot{\alpha}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$. Now you can multiply σ's and find

$$\sigma^\mu\bar{\sigma}^\nu = g^{\mu\nu} + 2\sigma^{\mu\nu} \tag{17}$$

where $\sigma^{\mu\nu} = 1/4(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$.

The usual discussion shows that

$$[\gamma^\mu(i\partial_\mu + eA_\mu) - m\mathbf{1}]\Psi^c = 0 \tag{18}$$

provided $\Psi^c = C\bar{\Psi}^T$ and $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$. Inspection of the γ's shows that we can choose $C = \omega\gamma^0\gamma^2$, where $|\omega| = 1$. Then

$$C = \omega \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \tag{19}$$

The usual Lorentz generators are

$$M^{\mu\nu} = i/4[\gamma^\mu, \gamma^\nu] \tag{20}$$

The chiral projections give

$$P_L M^{0i} P_L = -i/2 \begin{pmatrix} \sigma^i & 0 \\ 0 & 0 \end{pmatrix} \tag{21}$$

$$P_R M^{0i} P_R = i/2 \begin{pmatrix} 0 & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{22}$$

Under a small boost, the transformation matrix is

$$\mathbf{1} \mp \omega_j\sigma^j/2 \tag{23}$$

The transformation

$$\sigma^2[\mathbf{1} \mp \omega_j \sigma^j/2]^* \sigma^2 = [\mathbf{1} \pm \omega_j \sigma^j/2] \tag{24}$$

shows that $\sigma^2 \psi^*$ tranforms as $\bar{\chi}$ and $\sigma^2 \bar{\chi}^*$ tranforms as ψ. Since σ_2 is proportional to ϵ, they both raise or lower indices. Then we can take $\psi^*_\alpha = \bar{\psi}_{\dot{\alpha}}$ and $\bar{\chi}^{\dot{\alpha}\,*} = \chi^\alpha$. If the Dirac spinor in the Weyl basis, as given in 2. above, is

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix} \tag{25}$$

then the charge conjugate spinor $\Psi^c = C\bar{\Psi}^T$ is

$$\Psi^c = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{26}$$

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix} \tag{27}$$

The Majorana condition $\Psi = \Psi^c$ requires $\psi^\alpha = \chi_\alpha$, so that the Majorana spinor has the form

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \psi^{\dot{\alpha}} \end{pmatrix}. \tag{28}$$

General comments: Weyl fermions are massless, have definite chirality, transform covariantly, have two degrees of freedom, and correspond to $(1,0)$ or $(0,1)$ in the $SL(2,C)$ analysis.
Majorana fermions can have mass, transform covariantly, have two degrees of freedom, and correspond to a definite linear combination of $(1,0)$ and $(0,1)$, for example $(1,0) + \sigma_2(1,0)^*$, where $\sigma_2(1,0)^*$ transforms as $(0,1)$.
Dirac fermions have mass, transform covariantly, have four degrees of freedom, $(1,0) \oplus (0,1)$.