Physics 711, Symmetry Problems in Physics Fall 2005

Homework solutions for assignment 4

Solutions for assignment due 11/7/05

1. Because $R \in SO(3)$ is both unitary and real orthogonal its eigenvalues are $1, e^{i\phi}, e^{-i\phi}$. (Special cases can have eigenvalues $1, 1, 1$ or $1, -1, -1$, but these are exceptional.) The eigenvector for 1 is $\mathbf{e}_1 = \mathbf{n}$, where $\mathbf{n}$ is the axis of rotation, and the other two eigenvalues are complex conjugates of each other, $R\mathbf{e}_2 = e^{i\phi}\mathbf{e}_2$, $R\mathbf{e}_3 = e^{-i\phi}\mathbf{e}_3$. The angle ϕ is the angle of rotation around the $\mathbf{n}$ axis. The eigenvectors are orthonormal with the usual hermitian scalar product, $(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{e}_i^* \cdot \mathbf{e}_j = \delta_{ij}$. Since R is real, $R\mathbf{e}_2^* = e^{-i\phi}\mathbf{e}_2^*$, so $\mathbf{e}_3 = \mathbf{e}_2^*$. Let $\mathbf{e}_2 = 1/\sqrt{2}(\mathbf{a} + i\mathbf{b})$, $\mathbf{a}, \mathbf{b}$ real. Since $(\mathbf{e}_1, \mathbf{e}_2) = \mathbf{e}_1 \cdot (\mathbf{a} + i\mathbf{b}) = 0$, $\mathbf{e}_1 \cdot \mathbf{a} = 0, \mathbf{e}_1 \cdot \mathbf{b} = 0$. Since $(\mathbf{e}_2, \mathbf{e}_3) = \mathbf{a}^2 - \mathbf{b}^2 - 2i\mathbf{a} \cdot \mathbf{b} = 0$, $|\mathbf{a}| = |\mathbf{b}|$, and $\mathbf{a} \cdot \mathbf{b} = 0$. Then taking real and imaginary parts, $R\mathbf{e}_2 = R(\mathbf{a} + i\mathbf{b}) = (\cos\phi + i\sin\phi)(\mathbf{a} + i\mathbf{b})$ becomes

$$R \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}. \tag{1}$$

Let $\mathbf{e}_1 = \mathbf{x}_3, \mathbf{a} = \mathbf{x}_1, \mathbf{b} = \mathbf{x}_2$. Then

$$R \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix}. \tag{2}$$

2. By change of basis, can put R in the form above. Trace labels classes, so $1 + 2\cos\phi$ or ϕ labels classes.

Can also give a direct argument. The logic is that the class of a given g_0 in $SO(3)$ is the set of $g^{-1}\, g_0\, g$ where g is any element of $SO(3)$. This requires a calculation! It is easier to do the calculation in $SU(2)$ and use the homomorphism $SU(2) \to SO(3)$. [Exercise for the class: Show that classes are preserved under homomorphism.] Pick

$$g_0 = e^{i\sigma_z \phi/2} = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \tag{3}$$

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{4}$$

where $|\alpha|^2 + |\beta|^2 = 1$ as must hold for $SU(2)$. Then

$$g^{-1}\ g_0\ g = \begin{pmatrix} e^{i\phi/2}|\alpha|^2 + e^{-i\phi/2}|\beta|^2 & 2i\sin\phi/2\ \alpha^*\beta \\ 2i\sin\phi/2\ \alpha\beta^* & e^{-i\phi/2}|\alpha|^2 + e^{i\phi/2}|\beta|^2 \end{pmatrix} \tag{5}$$

Compare the result of (6) to the $SU(2)$ rotation by ϕ around axis $\mathbf{n}$,

$$g_n = e^{i\sigma\cdot n\phi/2} = \begin{pmatrix} \cos\phi/2\ \mathbf{1} + i\sin\phi/2 n_z & i\sin\phi/2(n_x - in_y) \\ i\sin\phi/2(n_x + in_y) & \cos\phi/2\ \mathbf{1} - i\sin\phi/2 n_z \end{pmatrix} \tag{6}$$

Here I leave to the class to show how to get this matrix from $exp(i\sigma \cdot n)$. Comparing 5 and 6 gives three equations

$$|\alpha|^2 e^{i\phi/2} + |\beta|^2 e^{-i\phi/2} = \cos\phi/2\ \mathbf{1} + i\sin\phi/2 \cdot n_z \tag{7}$$

$$\alpha\beta^* 2i\sin\phi/2 = i\sin\phi/2(n_x + in_y) \tag{8}$$

$$|\alpha|^2 e^{-i\phi/2} + |\beta|^2 e^{i\phi/2} = \cos\phi/2\ \mathbf{1} - i\sin\phi/2 \cdot n_z \tag{9}$$

I leave to the class to show that these equations are satisfied if

$$\alpha = \sqrt{\frac{1+n_z}{2}} e^{i\psi},\ \ \beta = (n_x - in_y)\sqrt{\frac{1}{2(1+n_z)}} e^{i\psi} \tag{10}$$

Thus the conjugation of g_0 by an arbitrary $g \in SU(2)$ gives a rotation by the *same* ϕ and thus $(\phi, \pm)$ labels classes in $SU(2)$. Note that $g \to -g$ is not an inner automorphism in $SU(2)$. Under the homomorphism $(\phi, \pm) \to \phi$.

3. From 1.,

$$R\begin{pmatrix} \frac{1}{\sqrt{2}}(\mathbf{x}_1 + i\mathbf{x}_2) \\ \frac{1}{\sqrt{2}}(\mathbf{x}_1 - i\mathbf{x}_2) \\ \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & e^{-i\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}(\mathbf{x}_1 + i\mathbf{x}_2) \\ \frac{1}{\sqrt{2}}(\mathbf{x}_1 - i\mathbf{x}_2) \\ \mathbf{x}_3 \end{pmatrix}. \tag{11}$$

Guess that

$$U = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \tag{12}$$

and check

$$\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \begin{pmatrix} \mathbf{x}_3 & \mathbf{x}_1 - i\mathbf{x}_2 \\ \mathbf{x}_1 + i\mathbf{x}_2 & -\mathbf{x}_3 \end{pmatrix} \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_3 & e^{-i\phi}(\mathbf{x}_1 - i\mathbf{x}_2) \\ e^{i\phi}(\mathbf{x}_1 + i\mathbf{x}_2) & -\mathbf{x}_3 \end{pmatrix} \tag{13}$$

4. The boost by hyperbolic angle ϕ in the z-direction is

$$\Lambda \begin{pmatrix} x^0 \\ x^3 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^3 \\ x^1 \\ x^2 \end{pmatrix}, \text{or } \Lambda x = x'. \tag{14}$$

The elements of $SL(2, C)$ that correspond to $\Lambda \in SO(1,3)$ obey $X' = AXA^\dagger$, $X = \sum_{\mu=0}^{3} x^\mu \sigma^\mu$. Need to find $A \ni X' = \sum_{\mu=0}^{3} (\Lambda x)^\mu \sigma^\mu$ if $X = \sum_{\mu=0}^{3} x^\mu \sigma^\mu$. Guess that

$$A = \pm \begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix}. \tag{15}$$

and check

$$\begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} = \begin{pmatrix} e^{\phi}(x^0 + x^3) & x^1 - ix^2 \\ x^1 + ix^2 & e^{-\phi}(x^0 - x^3) \end{pmatrix}, \tag{16}$$

which gives the correct Lorentz transformation.