Physics 711, Symmetry Problems in Physics Fall 2005

Homework

Solutions for assignment due 10/18/05

S_5

Label irreducibles by the number of horizontal boxes in each row, $(\lambda_1, \lambda_2, ...)$, with $\lambda_i \geq \lambda_j, i > j$.

Label classes by the number n_i of cycles of length ν_i, $(\nu_i^{n_i})$.

Both cases correspond to partitions. There are equal numbers of inequivalent irreducibles and of distinct classes.

Class α	*No. of elements* k_α	*Irreducible* a	*Dim* n_a
$1^5!$	$\frac{5!}{1^5 5!} = 1$	(5)	1
$1^3 2$	$\frac{5!}{1^3 3! 2^1 1!} = 10$	$(4,1)$	$\frac{5!}{1\cdot 2\cdot 3\cdot 5\cdot 1} = 4$
12^2	$\frac{5!}{1 1! 2^2 2!} = 15$	$(3,2)$	$\frac{5!}{1\cdot 3\cdot 4\cdot 1\cdot 2} = 4$
$1^2 3$	$\frac{5!}{1^2 ! 3^1 1!} = 20$	$(3,1,1)$	$\frac{5!}{1\cdot 2\cdot 5\cdot 2\cdot 1} = 6$
23	$\frac{5!}{2^1 1! 3^1 1!} = 20$	$(2,2,1)$	$\frac{5!}{2\cdot 4\cdot 1\cdot 3\cdot 1} = 5$
1^4	$\frac{5!}{1^1 ! 4^1 1!} = 30$	$(2,1,1,1)$	$\frac{5!}{1\cdot 5\cdot 3\cdot 2\cdot 1} = 4$
5	$\frac{5!}{5! 1!} = 24$	$(1,1,1,1,1)$	$\frac{5!}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 1$

$\sum_\alpha k_\alpha = 1 + 10 + 15 + 20 + 20 + 30 + 24 = 120 = 5!$

$\sum_a n_a^2 = 1^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 1^2 = 120 = 5!$

A cycle $(12...n) \equiv n - 1$ transpositions, so $\sigma(n - cycle) = (-1)^{n-1}$.

The character table is a 7x7 array $\chi_{a\alpha}$. Label the irreducibles and classes by integers 1, 2, ..., 7 in the order given above to save space. Then $\chi_{1,\alpha} = 1$, $\chi_{7,\alpha} = (1, -1, 1, 1, -1, -1, 1)$. $\chi_{a,e} = n_a$. Because irreducibles 2 and 6 are conjugate, $\chi_{2,\alpha}$ and $\chi_{6,\alpha}$ are related by the same pattern as the signs of $\chi_{7,\alpha}$. The same is true of irreducibles 3 and 5, so $\chi_{3,\alpha}$ and $\chi_{5,\alpha}$ also are related by the same pattern as the signs of $\chi_{7,\alpha}$. Irreducible 4 is self-conjugate so $\chi_{4,2} = \chi_{4,5} = \chi_{4,6} = 0$.

Now consider the defining rep D_{def} that acts on the integers $(1, 2, 3, 4, 5)$.

$D_{def} = D_1 \oplus D_2$, and $\chi_{def,\alpha} = \chi_{1\alpha} + \chi_{2\alpha}$. $\chi_{def,\alpha}(perm) =$ no. of fixed point of the perm. $chi_{1\alpha} = 1$, so $\chi_{2\alpha}(perm) =$ no. of fixed points -1. Thus

$\chi_{2\alpha} = 2, 0, 1, -1, 0, -1$ for $\alpha = 2, 3, 4, 5, 6, 7$, respectively; $\chi_{6\alpha} = -2, 0, 1, 1, 0, -1$ for $\alpha = 2, 3, 4, 5, 6, 7$, respectively.

Next consider the product of the defining representation with itself.
$D_{def} \otimes D_{def} = (D_1 \oplus D_2) \otimes (D_1 \oplus D_2) = 2D_1 \oplus 3D_2 \oplus D_3 \oplus D_4 =$
$[2D_1 \oplus 2D_2 \oplus D_3]_s \oplus [D_2 \oplus D_4]_a = D_{15,s} \oplus D_{10,a}$, where the subscripts s and a denote the symmetric and antisymmetric parts of the product of the defining representation and the reduction of the product and the separation into symmetric and antisymmetric parts uses the Young patterns and the intuition that products of more symmetric Young patterns decompose into more symmetric Young patterns and the analogous decomposition for more antisymmetric Young patterns. Thus $D_{15,s} = 2D_1 \oplus 2D_2 \oplus D_3$ and $D_{10,a} = D_2 \oplus D_4$, and the corresponding relations hold for the characters, $\chi_{15,s} = 2\chi_1 + 2\chi_2 + \chi_3$ and $\chi_{10,a} = \chi_2 + \chi_4$. Since we already know $\chi_{1,\alpha}$ and $\chi_{2,\alpha}$ we can find $\chi_{3,\alpha}$ and $\chi_{4,\alpha}$ if we can calculate $\chi_{15,\alpha}$ and $\chi_{10,\alpha}$.

To calculate each of these we choose a basis of either symmetric or antisymmetric pairs of the numbers 1 to 5 and act on each member of the basis with any permutation from each of the seven classes; the corresponding character is the number of fixed points under the action of the given permutation. We use permutations on named objects for this calculation. (Normalization of the states does not matter here.) For example, the perm (12) from class 2 acting on $D_{15,s}$ has fixed points for (12) and the six states (33), $(34 + 43)$, $\cdots$ (55), so $\chi_{15,2} = 7$. The results are $\chi_{15,\alpha} = 15, 7, 3, 3, 1, 1, 0$ and $\chi_{10,\alpha} = 10, 2, -2, 1, -1, 0, 0$. Now we get row three also row five (from D_{10}) and row four (from from D_5) which completes the character table.

$Rep \vert Class$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	4	2	0	1	-1	0	-1
3	5	1	1	-1	1	-1	0
4	6	0	-2	0	0	0	1
5	5	-1	1	-1	-1	1	0
6	4	-2	0	1	1	0	-1
7	1	-1	1	1	-1	-1	1

To get normalized characters multiply the columns by $\sqrt{k_\alpha/120} = 1/\sqrt{120}, 1/\sqrt{12}, 1/\sqrt{8}, 1/\sqrt{6}, 1/\sqrt{6}, 1/2, 1/\sqrt{5}$ respectively.