Homework 9 Solutions

9.1 - Commutation relations

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \left[\varepsilon_r^\mu(k)\, a_r(k)\, e^{-ik\cdot x} + \varepsilon_r^{*\mu}(k)\, a_r^\dagger(k)\, e^{ik\cdot x} \right]$$

$$\Pi^\nu(x) = -\dot{A}^\nu = i\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3} \left[\varepsilon_r^\nu(k)\, \omega_k\, a_r(k)\, e^{-ik\cdot x} - \varepsilon_r^{*\nu}(k)\, \omega_k\, a_r^\dagger(k)\, e^{ik\cdot x} \right]$$

(In Feynman gauge)

$$\Rightarrow \left[A^\mu(x), \Pi^\nu(y) \right]_- = i\int \frac{d^3k\, d^3k'}{(2\pi)^3 2\sqrt{\omega_k \omega_{k'}}} \sum_{r,r'=0}^{3} \Big\{ \left[a_r(k), a_{r'}^\dagger(k') \right]_- \left(-\varepsilon_r^\mu(k)\, \varepsilon_{r'}^{*\nu}(k')\, \omega_{k'}\, e^{ik'\cdot y - ik\cdot x}\right)$$

$$+ \left[a_r^\dagger(k), a_{r'}(k') \right]_- \left(\varepsilon_r^{*\mu}(k)\, \varepsilon_{r'}^\nu(k')\, \omega_{k'}\, e^{ik\cdot x - ik'\cdot y} \right) \Big\}$$

use $\left[a_r(k), a_{r'}^\dagger(k') \right]_- = \zeta_r\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')$

$$\Rightarrow \left[A^\mu(x), \Pi^\nu(y) \right]_- = i\int \frac{d^3k\, d^3k'}{(2\pi)^3 2\sqrt{\omega_k \omega_{k'}}} \sum_{r,r'=0}^{3} \Big\{ -\varepsilon_r^\mu(k)\, \varepsilon_{r'}^{*\nu}(k')\, \omega_{k'}\, e^{ik'\cdot y - ik\cdot x}\, \zeta_r\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')$$

$$- \zeta_r\, \delta_{rr'}\, \delta^3(\vec{k}-\vec{k}')\, \varepsilon_r^{*\mu}(k)\, \varepsilon_{r'}^\nu(k')\, \omega_{k'}\, e^{ik\cdot x - ik'\cdot y} \Big\}$$

$$= (-i) \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=0}^{3} \left[\zeta_r\, \varepsilon_r^\mu(k)\, \varepsilon_r^{*\nu}(k)\, \omega_k\, e^{ik\cdot(y-x)} + \zeta_r\, \varepsilon_r^{*\mu}(k)\, \varepsilon_r^\nu(k)\, \omega_k\, e^{ik\cdot(x-y)} \right]$$

use the completeness relation $\sum_{r=0}^{3} \zeta_r\, \varepsilon_r^\mu\, \varepsilon_r^{*\nu} = -g^{\mu\nu}$

$\Rightarrow$ The equal time commutator:

$$\left[A_\mu(\vec{x},t), \Pi^\nu(\vec{y},t) \right]_- = i\, \delta_\mu^\nu \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = i\, \delta_\mu^\nu\, \delta^3(\vec{x}-\vec{y}) \quad \square.$$

9.2 - Hamiltonian and momentum

Part (i)

The Lagrangian with a R_ξ gauge fixing term as the following stress energy tensor

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}\left(1-\frac{1}{\xi}\right)(\partial_\mu A^\mu)^2$$

$$T^{0\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\lambda)}\partial^\mu A^\lambda - g^{0\mu}\mathcal{L}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A^\lambda)} = -\dot{A}_\lambda + \left(1-\frac{1}{\xi}\right)\delta_0^{\ \lambda}(\partial_\nu A^\nu)$$

$$\Rightarrow T^{0\mu} = -\dot{A}_\lambda \partial^\mu A^\lambda + \left(1-\frac{1}{\xi}\right)(\partial^\mu A^0)(\partial_\nu A^\nu) + \frac{g^{0\mu}}{2}(\partial_\lambda A_\nu)(\partial^\lambda A^\nu)$$
$$- \frac{g^{0\mu}}{2}\left(1-\frac{1}{\xi}\right)(\partial_\nu A^\nu)^2$$

The Feynman-t'Hooft gauge is obtained by taking $\xi = 1$.

$$\Rightarrow T^{0\mu} = -\dot{A}_\lambda \partial^\mu A^\lambda + \frac{1}{2}g^{0\mu}(\partial_\lambda A_\nu)(\partial^\lambda A^\nu)$$

Total 4-momentum of the free photon field.

$$P^\mu = \int d^3x\, T^{0\mu} = \int d^3x \left[-\dot{A}_\lambda \partial^\mu A^\lambda + \frac{1}{2}g^{0\mu}(\partial_\lambda A_\nu)(\partial^\lambda A^\nu)\right] \quad \left(\text{This is independent of } \xi\right)$$

First consider P^0:

$$P^0 = \int d^3x \left[-\dot{A}_\lambda \dot{A}^\lambda + \frac{1}{2}(\partial_\lambda A_\nu)(\partial^\lambda A^\nu)\right]$$

Recall
$$A_\mu = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{r=0}^{3}\left[\varepsilon_r^\mu(k)\, a_r(k)\, e^{-ik\cdot x} + \varepsilon_r^{*\mu}(k)\, a_r^\dagger(k)\, e^{ik\cdot x}\right]$$

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$$\Rightarrow \int d^3x \, (-\dot{A}_\lambda \dot{A}^\lambda)$$

$$= -\int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\sqrt{\omega_k \omega_{k'}}} (-\omega_k \omega_{k'}) \sum_{r,r'=0}^{3} \Big\{ \varepsilon_r^\lambda(k) a_r(k) a_{r'}(k') \varepsilon_{r',\lambda}(k') e^{-i(k'+k)\cdot x} + \varepsilon_r^{*\lambda}(k) a_r^\dagger(k) a_{r'}^\dagger(k') \varepsilon_{r',\lambda}^* e^{i(k+k')\cdot x}$$

$$- \varepsilon_r^\lambda(k) a_r(k) \varepsilon_{r',\lambda}^*(k') a_{r'}^\dagger(k') e^{-i(k-k')\cdot x} - \varepsilon_r^{*\lambda}(k) a_r^\dagger(k) \varepsilon_{r',\lambda}(k') a_{r'}(k') e^{i(k-k')\cdot x} \Big\}$$

Carry out the $\int d^3x$ integral

$$= \int \frac{d^3k}{2} \omega_k \sum_{r,r'=0}^{3} \Big\{ \varepsilon_r^\lambda(k) a_r(k) a_{r'}(-k) \varepsilon_{r',\lambda}(-k) e^{-i2\omega_k t} + \varepsilon_r^{*\lambda}(k) a_r^\dagger(k) a_{r'}^\dagger(-k) \varepsilon_{r',\lambda}^*(-k) e^{i2\omega_k t}$$

$$- \varepsilon_r^\lambda(k) a_r(k) \varepsilon_{r',\lambda}^*(k) a_{r'}^\dagger(k) - \varepsilon_r^{*\lambda}(k) a_r^\dagger(k) \varepsilon_{r',\lambda}(k) a_{r'}(k) \Big\}$$

$$\frac{1}{2} \int d^3x \, (\partial_\lambda A_\nu)(\partial^\lambda A^\nu)$$

$$= \frac{1}{2} \int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\sqrt{\omega_k \omega_{k'}}} (-k_\lambda k'^\lambda) \sum_{r,r'=0}^{3} \Big\{ \varepsilon_r(k) a_r(k) a_{r'}(k') \varepsilon_{r',\lambda}(k') e^{-i(k'+k)\cdot x}$$

$$+ \varepsilon_r^{*\lambda}(k) a_r^\dagger(k) a_{r'}^\dagger(k') \varepsilon_{r',\lambda}^* e^{i(k+k')\cdot x} - \varepsilon_r^\lambda(k) a_r(k) \varepsilon_{r',\lambda}^*(k') a_{r'}^\dagger(k') e^{-i(k-k')\cdot x} - \varepsilon_r^{*\lambda}(k) a_r^\dagger(k) \varepsilon_{r',\lambda}(k') a_{r'}(k') e^{i(k-k')\cdot x} \Big\}$$

(note that $k^2=0$ and $k\cdot\tilde{k} = 2\omega_k^2$ $(\tilde{k}^\mu = (\omega_k, -\vec{k}))$)

$$= -\frac{1}{2} \int \frac{d^3k}{2} \omega_k \sum_{r,r'=0}^{3} \Big\{ 2\varepsilon_r^\lambda(k) a_r(k) a_{r'}(-k) \varepsilon_{r',\lambda}(-k) e^{-i2\omega_k t} + 2\varepsilon_r^{*\lambda}(k) a_r^\dagger(k) a_{r'}^\dagger(-k) \varepsilon_{r',\lambda}^*(-k) e^{i2\omega_k t} \Big\}$$

Adding these two terms

$$\Rightarrow P^0 = -\int \frac{d^3k}{2} \omega_k \sum_{r,r'=0}^{3} \Big[\varepsilon_r^\lambda(k) \varepsilon_{r',\lambda}^*(k) a_r(k) a_{r'}^\dagger(k) + \varepsilon_r^{*\lambda}(k) \varepsilon_{r',\lambda}(k) a_r^\dagger(k) a_{r'}(k) \Big]$$

(use $\varepsilon_r^\lambda(k) \varepsilon_{r',\lambda}^*(k) = -\zeta_r \delta_{r,r'}$)

$$= \int \frac{d^3k}{2} \omega_k \sum_{r=0}^{3} \big[\zeta_r a_r(k) a_r^\dagger(k) + \zeta_r a_r^\dagger(k) a_r(k) \big]$$

After normal ordering $\Rightarrow$ $:P^0: = \int d^3k \, \omega_k \sum_{r=0}^{3} [\zeta_r a_r^\dagger(k) a_r(k)]$

P^0 is the Hamiltonian.

We can choose a frame such that $a_0(k)|\Psi\rangle = a_3(k)|\Psi\rangle$ (5)

Then $\boxed{:P^0: = \int d^3k \sum_{r=1}^{2} a_r^\dagger(k) a_r(k)}$, we see that it only depends on the transverse modes.

Part (ii)

Next, we consider $P^i = \int d^3x \left(-\dot{A}_\lambda \partial^i A^\lambda \right)$

$$= + \int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^3 2\sqrt{\omega_k \omega_{k'}}} (+\omega_k k'^i) \sum_{r,r'=0}^{3} \Big\{ \varepsilon_r^\lambda(k)\, \varepsilon_{r',\lambda}(k')\, a_r(k)\, a_{r'}(k')\, e^{-i(k+k')\cdot x}$$

$$+ \varepsilon^{*\lambda}_r(k)\, \varepsilon^*_{r',\lambda}(k')\, a_r^\dagger(k)\, a_{r'}^\dagger(k')\, e^{i(k+k')\cdot x} - \varepsilon_r^\lambda(k)\, \varepsilon^*_{r',\lambda}(k')\, a_r(k)\, a_{r'}^\dagger(k')\, e^{-i(k-k')\cdot x} - \varepsilon^{*\lambda}_r(k)\, a_r^\dagger(k)\, \varepsilon_{r',\lambda}(k')\, a_{r'}(k')\, e^{i(\ldots}$$

$$= + \int \frac{d^3k}{2} (-k^i) \sum_{r,r'=0}^{3} \Big\{ \varepsilon_r^\lambda(k)\, \varepsilon_{r',\lambda}(-k)\, a_r(k)\, a_{r'}(-k)\, e^{-2i\omega_k t} + \varepsilon^{*\lambda}_r(k)\, \varepsilon^*_{r',\lambda}(-k)\, a_r^\dagger(k)\, a_{r'}^\dagger(-k)\, e^{2i\omega_k t}$$

$$+ \varepsilon_r^\lambda(k)\, \varepsilon^*_{r',\lambda}(k)\, a_r(k)\, a_r^\dagger(k) + \varepsilon^{*\lambda}_r(k)\, \varepsilon_{r,\lambda}(k)\, a_r^\dagger(k)\, a_r(k) \Big\}$$

The first two terms vanish since they are odd under $k^i \to -k^i$

$$\Rightarrow P^i = + \int \frac{d^3k}{2} k^i \sum_{r=0}^{3} \left[\zeta_r a_r(k) a_r^\dagger(k) + \zeta_r a_r^\dagger(k) a_r(k) \right]$$

$$= \int d^3k \; k^i \sum_{r=0}^{3} \left[\zeta_r a_r^\dagger(k) a_r(k) \right]$$

Choose the frame such that $a_0(k)|\psi\rangle = a_3(k)|\psi\rangle = 0$

$$\Rightarrow \boxed{P^i = \int d^3k \; k^i \sum_{r=1}^{2} a_r^\dagger(k) a_r(k)}$$

We see that P^i also only depends on the transverse modes. $\square$.

9.3 - Residual gauge invariance

Part (i)

Since $|\Psi_T\rangle$ contains only transverse polarizations, the longitudinal and time-like annihilation operators kill the state.

$$[a_3(\vec{k}) - a_0(\vec{k})]\,|\Psi_T\rangle = 0 \tag{1}$$

Part (ii)

$$\begin{aligned}
[a_3(\vec{k}) - a_0(\vec{k})]\,|\Psi'_T\rangle &= [a_3(\vec{k}) - a_0(\vec{k})][1 + c[a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})]]\,|\Psi_T\rangle && (2)\\
&= c[a_3(\vec{k}) - a_0(\vec{k})][a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})]\,|\Psi_T\rangle && (3)\\
&= c[a_3(\vec{k})a_3^\dagger(\vec{k}) + a_0(\vec{k})a_0^\dagger(\vec{k})]\,|\Psi_T\rangle && (4)\\
&= c[\zeta_3\delta^3(0) + a_3^\dagger(\vec{k})a_3(\vec{k}) + \zeta_0\gamma^3(0) + a_0^\dagger(\vec{k})a_0(\vec{k})]\,|\Psi_T\rangle && (5)\\
&= 0 && (6)
\end{aligned}$$

where we have used the commutation relation,

$$[a_r(k), a_s^\dagger(k')] = \zeta_r \delta_{rs} \delta^3(\vec{k} - \vec{k}') \tag{7}$$

Part (iii)

$$\begin{aligned}
\langle \Psi'_T | A^\mu(x) | \Psi'_T \rangle &= \langle \Psi_T | [1 + c^*[a_3(\vec{k}) - a_0(\vec{k})]] A^\mu [1 + c[a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})]] | \Psi_T \rangle &(8)\\
&= \langle \Psi_T | A^\mu | \Psi_T \rangle &(9)\\
&\quad + c^* \langle \Psi_T | [a_3(\vec{k}) - a_0(\vec{k})] A^\mu | \Psi_T \rangle &(10)\\
&\quad + c \langle \Psi_T | A^\mu [a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})] | \Psi_T \rangle &(11)\\
&\quad + |c|^2 \langle \Psi_T | [a_3(\vec{k}) - a_0(\vec{k})] A^\mu [a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})] | \Psi_T \rangle &(12)
\end{aligned}$$

It is clear that the fourth term doesn't contribute, because it has an odd number of creation/annihilation operators sandwiched between identical states. Let us look at one of the "mixed" terms,

$$\begin{aligned}
\langle \Psi_T | A^\mu [a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})] | \Psi_T \rangle &= \langle \Psi_T | \int \frac{d^3q}{(2\pi)^{\frac{3}{2}} \sqrt{2\omega_q}} \sum_{r=0}^{3} \epsilon_r^\mu(q) a_r(q) e^{-iq\cdot x} [a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})] | \Psi_T \rangle \\
&\quad + \langle \Psi_T | \int \frac{d^3q}{(2\pi)^{\frac{3}{2}} \sqrt{2\omega_q}} \sum_{r=0}^{3} \epsilon_r^{\mu *}(q) a_r^\dagger(q) e^{iq\cdot x} [a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})] | \Psi_T \rangle &(13)\\
&= \int \frac{d^3q}{(2\pi)^{\frac{3}{2}} \sqrt{2\omega_q}} \langle \Psi_T | e^{-iq\cdot x} [\epsilon_3^\mu a_3(\vec{q}) a_3^\dagger(\vec{k}) - \epsilon_0^\mu a_0(\vec{q}) a_0^\dagger(\vec{k})] | \Psi_T \rangle &(14)\\
&= \int \frac{d^3q}{(2\pi)^{\frac{3}{2}} \sqrt{2\omega_q}} \langle \Psi_T | e^{-iq\cdot x} [\epsilon_3^\mu \zeta_3 \delta^3(\vec{q} - \vec{k}) - \epsilon_0^\mu \zeta_0 \delta^3(\vec{q} - \vec{k})] | \Psi_T \rangle &(15)\\
&= \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ik\cdot x} (\epsilon_3^\mu + \epsilon_0^\mu) &(16)
\end{aligned}$$

Now, we can choose the time-like polarization in some direction n^μ. Then, the longitudinal polarization ($k^2 = 0$ in equation 8.50 of L&P),

$$\epsilon_0 = n^\mu \qquad \epsilon_3 = \frac{1}{\sqrt{k \cdot n}} k^\mu - n^\mu \tag{17}$$

Therefore

$$c \langle \Psi_T | A^\mu [a_3^\dagger(\vec{k}) - a_0^\dagger(\vec{k})] | \Psi_T \rangle = c \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \frac{1}{\sqrt{k \cdot n}} k^\mu e^{-ik\cdot x} \sim \partial^\mu (e^{-ik\cdot x}) \tag{18}$$

We can treat the other mixed term above in exactly the same way, and the result can be guessed to be the complex conjugate of the above,

$$c^* \langle \Psi_T | [a_3(\vec{k}) - a_0(\vec{k})] A^\mu | \Psi_T \rangle = c^* \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \frac{1}{\sqrt{k \cdot n}} k^\mu e^{ik\cdot x} \qquad \sim \partial^\mu (e^{ik\cdot x}) \tag{19}$$

Finally,

$$\langle \Psi'_T | A^\mu(x) | \Psi'_T \rangle = \langle \Psi_T | A^\mu | \Psi_T \rangle + \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} \frac{1}{\sqrt{k \cdot n}} (ic\partial^\mu(e^{-ik\cdot x}) - ic^* \partial^\mu (e^{ik\cdot x})) \tag{20}$$

$$= \langle \Psi_T | A^\mu + \partial^\mu \theta(x) | \Psi_T \rangle \tag{21}$$

where $\theta \sim Re(ce^{-ik\cdot x})$.

9.4 - Photon propagator

Assume initially for simplicity that $x^0 > x'^0$. The propagator,

$$\langle 0 | T [A^\mu(x) A^\nu(x')] | 0 \rangle = \langle 0 | \Bigg[\int \frac{d^3q}{(2\pi)^{\frac{3}{2}} \sqrt{2\omega_q}} \sum_{r=0}^{3} \epsilon_r^\mu(q) a_r(q) e^{-iq\cdot x} + \epsilon_r^{\mu *}(q) a_r^\dagger(q) e^{iq\cdot x}$$
$$\int \frac{d^3k}{(2\pi)^{\frac{3}{2}} \sqrt{2\omega_k}} \sum_{s=0}^{3} \epsilon_s^\nu(k) a_s(k) e^{-ik\cdot x'} + \epsilon_s^{\nu *}(k) a_s^\dagger(k) e^{ik\cdot x'} \Bigg] |0\rangle \tag{22}$$

$$= \int \frac{d^3q d^3k}{(2\pi)^3 \sqrt{4\omega_k \omega_q}} \sum_{r=0}^{3} \sum_{s=0}^{3} \epsilon_r^\mu(q) e^{-iq\cdot x} \epsilon_s^{\nu *}(k) e^{ik\cdot x'} \langle 0 | T \left[a_r(q) a_s^\dagger(k) \right] | 0 \rangle \tag{23}$$

$$= \int \frac{d^3q d^3k}{(2\pi)^3 \sqrt{4\omega_k \omega_q}} \sum_{r=0}^{3} \sum_{s=0}^{3} \epsilon_r^\mu(q) e^{-iq\cdot x} \epsilon_s^{\nu *}(k) e^{ik\cdot x'} \zeta_r \delta_{rs} \delta^3(\vec{k} - \vec{q}) \tag{24}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{r=0}^{3} e^{ik\cdot(x-x')} \epsilon_r^\mu(k) \epsilon_s^{\nu *}(k) \zeta_r \tag{25}$$

Using

$$\sum_{r=0}^{3} \zeta_r \epsilon_r^\mu \epsilon_r^{*\nu} = -g^{\mu\nu} \tag{26}$$

We will get the same results with x, x' interchanged when $x^0 < x'^0$. Therefore, the full Feynman propagator is,

$$\langle 0 | T [A^\mu(x) A^\nu(x')] | 0 \rangle = -\frac{g^{\mu\nu}}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} e^{ik\cdot(x-x')} \theta(x^0 - x'^0) + e^{-ik\cdot(x-x')} \theta(x'^0 - x^0) \tag{27}$$

$$= -\frac{g^{\mu\nu}}{(2\pi)^4} \int d^4k \frac{e^{-ik\cdot(x-x')}}{k^2 + i\epsilon} \tag{28}$$

The last step is the standard Feymnan prescription (described for example in P&S section 2.4). The $+i\epsilon$ in the denominator gives the propagator the correct pole structure so that the dk^0 integral (equation 28)along the real line yields the time-ordered form in equation 27.

9.5 - Gauge invariance for charged scalar field

$$\mathcal{L} = (D^\mu\phi)^\dagger(D_\mu\phi) - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Gauge transformation $\phi \to \phi' = e^{-ieQ\theta}\phi$

$$A_\mu \to A_\mu' = A_\mu + \partial_\mu\theta$$

$$\Rightarrow D_\mu\phi = (\partial_\mu + ieQA_\mu)\phi \longrightarrow \left[\partial_\mu + ieQ(A_\mu + \partial_\mu\theta)\right](e^{-ieQ\theta}\phi)$$

$$= e^{-ieQ\theta}[\partial_\mu\phi - \cancel{ieQ\partial_\mu\theta\phi} + ieQA_\mu\phi + \cancel{ieQ\partial_\mu\theta\phi}]$$

$$= e^{-ieQ\theta}(D_\mu\phi)$$

Therefore $(D^\mu\phi)^\dagger(D_\mu\phi) \to (D^\mu\phi)^\dagger e^{ieQ\theta}\cdot e^{-ieQ\theta}(D_\mu\phi)$

$$= (D^\mu\phi)^\dagger(D_\mu\phi)$$

$$-m^2\phi^\dagger\phi \to -m^2\phi^\dagger e^{ieQ\theta}e^{-ieQ\theta}\phi = -m^2\phi^\dagger\phi.$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow \partial_\mu(A_\nu + \partial_\nu\theta) - \partial_\nu(A_\mu + \partial_\mu\theta)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$\Rightarrow -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \to -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Put them altogether $\Rightarrow$ $\mathcal{L} \to \mathcal{L}$, the Lagrangian is invariant under gauge transformation. □

9.6 - Covariant derivative for fermion field

$$\psi \to \psi' = e^{-ieQ\theta}\psi$$

$$\Rightarrow D_\mu'\psi' = \left[\partial_\mu + ieQ(A_\mu + \partial_\mu\theta)\right]e^{-ieQ\theta}\psi$$

$$= e^{-ieQ\theta}[\partial_\mu\psi - \cancel{ieQ\partial_\mu\theta\psi} + ieQA_\mu\psi + \cancel{ieQ\partial_\mu\theta\psi}]$$

$$= e^{-ieQ\theta}D_\mu\psi$$

9.7 - Axial-vector current

Part (i)

Under an infinitesimal transformation,

$$\psi \to e^{i\alpha\gamma^5}\psi \tag{29}$$

the shift in the field is given by,

$$\alpha\delta\psi = i\alpha\gamma^5\psi \tag{30}$$

Therefore, the Noether current corresponding to this tranformation is,

$$\begin{align} j^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi \tag{31}\\ &= -\bar{\psi}\gamma^\mu\gamma^5\psi \tag{32}\end{align}$$

Part (ii)

Under the transformation the kinetic term,

$$\begin{align} \bar{\psi}(i\partial_\mu\gamma^\mu)\psi &\to \overline{e^{i\alpha\gamma^5}\psi}(i\partial_\mu\gamma^\mu)e^{i\alpha\gamma^5}\psi \tag{33}\\ &= (e^{i\alpha\gamma^5}\psi)^\dagger\gamma^0(i\partial_\mu\gamma^\mu)e^{i\alpha\gamma^5}\psi \tag{34}\\ &= (\psi)^\dagger e^{-i\alpha\gamma^5}\gamma^0(i\partial_\mu\gamma^\mu)e^{i\alpha\gamma^5}\psi \tag{35}\\ &= (\psi)^\dagger\gamma^0 e^{i\alpha\gamma^5}(i\partial_\mu\gamma^\mu)e^{i\alpha\gamma^5}\psi \tag{36}\\ &= (\psi)^\dagger\gamma^0(i\partial_\mu\gamma^\mu)e^{i\alpha\gamma^5}e^{i\alpha\gamma^5}\psi \tag{37}\\ &= \bar{\psi}(i\partial_\mu\gamma^\mu)\psi \tag{38}\end{align}$$

where we have used the following fact.

$$e^{i\alpha\gamma^5}\gamma^\mu = \gamma^\mu e^{-i\alpha\gamma^5} \tag{39}$$

Therefore, the kinetic term in the Lagrangian is invariant under the transformation. What about the mass term?

$$\begin{align} m\bar{\psi}\psi &\to m\,\overline{e^{i\alpha\gamma^5}\psi}e^{i\alpha\gamma^5}\psi \tag{40}\\ &= m\,(\psi)^\dagger\gamma^0 e^{i\alpha\gamma^5}e^{i\alpha\gamma^5}\psi \tag{41}\\ &\neq \bar{\psi}(i\partial_\mu\gamma^\mu)\psi \tag{42}\end{align}$$

The mass term is not invariant under this transformation.

Part (iii)

In the limit $m \to 0$, the total Lagrangian is invariant under the transformation.

Part (iv)

The divergence of the current,

$$\partial_\mu j^\mu = \partial_\mu \left[\bar{\psi}\gamma^\mu\gamma^5\psi\right] \tag{43}$$
$$= (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi) \tag{44}$$
$$= (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu(\partial_\mu\psi) \tag{45}$$

Using the Dirac equation ,

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{46}$$

and its conjugate version,

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0 \tag{47}$$
$$\Rightarrow - i\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - m\psi^\dagger = 0 \tag{48}$$
$$\Rightarrow - i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 - m\psi^\dagger = 0 \tag{49}$$
$$\Rightarrow - i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0 \tag{50}$$

We find the divergence is,

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu(\partial_\mu\psi) \tag{51}$$
$$= im\bar{\psi}\gamma^5\psi - \bar{\psi}\gamma^5(-im\psi) \tag{52}$$
$$= 2im\bar{\psi}\gamma^5\psi \tag{53}$$

Part (v)

The current is not conserved because it the transformation is not a symmetry of the Lagrangian. We found earlier that it does become a symmetry in the limit $m \to 0$, in which limit the divergence of the current also vanishes. This is expected from Noether's theorem.

Part (vi)

We need to choose a representation for the γ matrices. The most convenient one to use in this case is the chiral (or Weyl) basis.

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{54}$$
$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{55}$$

where $\sigma^\mu = (I, \vec{\sigma})$ and $\bar{\sigma^\mu} = (I, -\vec{\sigma})$.

We can now break up the field ψ into its two chiralities ψ_L and ψ_R,

$$\psi = (L + R)\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{56}$$

where $L, R = \frac{1}{2}(1 \pm \gamma^5)$ are the usual chirality projectors.

The transformation,

$$\psi \to e^{i\alpha\gamma^5}\psi \tag{57}$$

can be written as the transformation of each chiral fermion,

$$\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \to e^{i\alpha\gamma^5}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \tag{58}$$

$$= \begin{pmatrix}e^{i\alpha}\psi_L\\ e^{-i\alpha}\psi_R\end{pmatrix} \tag{59}$$

using the fact that γ^5 is diagonal in this basis. The transformation rotates the two chiralities in opposite directions (in contrast with a "vector" transformation, $\psi \to \exp(i\alpha)\psi$, which rotates the two chiralities in the same direction).

Part (vii)

The Lagrangian in the Weyl basis is given by,

$$\mathcal{L} = \begin{pmatrix}\psi_L^\dagger & \psi_R^\dagger\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\left[i\partial_\mu\begin{pmatrix}0 & \sigma^\mu\\ \bar{\sigma}^\mu & 0\end{pmatrix} - m\right]\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \tag{60}$$

$$= \begin{pmatrix}\psi_L^\dagger & \psi_R^\dagger\end{pmatrix}\left[i\partial_\mu\begin{pmatrix}\bar{\sigma}^\mu & 0\\ 0 & \sigma^\mu\end{pmatrix} - m\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}\right]\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} \tag{61}$$

We see that the kinetic term is diagonal in the chirality basis (i.e. it does not couple different chiralities). The mass term, however, is off diagonal, and couples the left-chirality fermion with the right-chirality fermion,

$$\mathcal{L} = \psi_L^\dagger i\partial_\mu\bar{\sigma^\mu}\psi_L + \psi_L^\dagger i\partial_\mu\sigma^\mu\psi_L - m(\psi_R^\dagger\psi_L + \psi_L^\dagger\psi_R) \tag{62}$$

Part (viii)

Thus, we can understand why the divergence of the current depended on the mass. The mass term couples the two chiralities, which does not allow us to perform independent (or in this case, opposite) rotations on ψ_L and ψ_R.

These independent rotations are called chiral rotations(for obvious reasons), and when they are a good symmetry of the Lagrangian, the Lagrangian is said to possess chiral symmetry. A (Dirac) mass term explicitly breaks the chiral symmetry.

Part (ix)

If the current is conserved (i.e. chiral symmetry is preserved), then there is no problem in coupling this current to an electromagnetic field. However, if there is a mass term which breaks chiral symmetry, a coupling of the form $j^\mu A_\mu$ will no longer be gauge-invariant.