Homework 8 Solutions

8.1 - Simplifying spinors

① prove $[\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')]^* = [\bar{v}_{s'}(\vec{p}') F^{\ddagger} u_s(\vec{p})]$

$$[\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')]^* = [\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')]^\dagger = [v_{s'}^\dagger(\vec{p}') F^\dagger \gamma_0 u_s(\vec{p})]$$

$$= [\bar{v}_{s'}(\vec{p}') \gamma_0 F^\dagger \gamma_0 u_s(\vec{p})] = [\bar{v}_{s'}(\vec{p}') F^{\ddagger} u_s(\vec{p})]$$

② prove that $\sum_{s,s'} |\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')|^2 = \mathrm{Tr}[(\not{p}+m) F (\not{p}'-m) F^{\ddagger}]$ ②

$$\sum_{s,s'} |\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')|^2$$

$$= \sum_{s,s'} [\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')][\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')]^*$$

$$= \sum_{s,s'} [\bar{u}_s(\vec{p}) F v_{s'}(\vec{p}')][\bar{v}_{s'}(\vec{p}') F^{\ddagger} u_s(\vec{p})]$$

$$= \mathrm{Tr}\left[\left(\sum_s u_s(\vec{p})\bar{u}_s(\vec{p})\right) F \left(\sum_{s'} v_{s'}(\vec{p}')\bar{v}_{s'}(\vec{p}')\right) F^{\ddagger}\right]$$

$$= \mathrm{Tr}[(\not{p}+m) F (\not{p}'-m) F^{\ddagger}]$$

8.2 - Decay of scalar particle into electron positron pair

8.2.1 Specific fermion helicities

Part (i)

The expression we want to evaluate can be rewritten as,

$$\sum_{s,s'} \left|\overline{R u_s(\vec{p})} v_{s'}(\vec{p}')\right|^2 = \sum_{s,s'} \left|u_s^\dagger(\vec{p}) R^\dagger \gamma^0 v_{s'}(\vec{p}')\right|^2 \tag{1}$$

$$= \sum_{s,s'} \left|u_s^\dagger(\vec{p}) \gamma^0\gamma^0 R^\dagger \gamma^0 v_{s'}(\vec{p}')\right|^2 \tag{2}$$

$$= \sum_{s,s'} \left|u_s^\dagger(\vec{p}) \gamma^0 R^{\ddagger} v_{s'}(\vec{p}')\right|^2 \tag{3}$$

$$= \sum_{s,s'} \left|\overline{u_s(\vec{p})} R^{\ddagger} v_{s'}(\vec{p}')\right|^2 \tag{4}$$

(i) $L = \frac{1}{2}(1-\gamma_5)$; $R = \frac{1}{2}(1+\gamma_5)$

$\Rightarrow \bar{L} = \frac{1}{2}\gamma_0(1-\gamma_5^\dagger)\gamma_0 = \frac{1}{2}(1+\gamma_5) = R$

$\bar{R} = \frac{1}{2}\gamma_0(1+\gamma_5^\dagger)\gamma_0 = \frac{1}{2}(1-\gamma_5) = L$

$$\Rightarrow \sum_{s,s'} |\bar{u}_s(\vec{p})\, L\, v_{s'}(\vec{p}')|^2 = \mathrm{Tr}[(\not{p}+m) L (\not{p}'-m) R]$$

$$= \frac{1}{4}\Big\{\mathrm{Tr}[\not{p}(1-\gamma_5)\not{p}'(1+\gamma_5)] + m\,\mathrm{Tr}[(1-\gamma_5)\not{p}'(1+\gamma_5)]^{\to 0}$$

$$+ m\,\mathrm{Tr}[\not{p}(1+\gamma_5)(-m)(1+\gamma_5)]^{\to 0} - m^2\,\mathrm{Tr}[(1-\gamma_5)(1+\gamma_5)]^{\to 0}\Big\}$$

note that trace of odd number of gamma matrices is zero!

$$\Rightarrow \sum_{s,s'} |\bar{u}_s(\vec{p})\, L\, v_{s'}(\vec{p}')|^2 = \frac{1}{4}\mathrm{Tr}[\not{p}\not{p}' - \not{p}\gamma_5\not{p}' + \not{p}\not{p}'\gamma_5 - \not{p}\gamma_5\not{p}'\gamma_5]$$

$$= \frac{1}{2}\mathrm{Tr}[\not{p}\not{p}' - \not{p}\gamma_5\not{p}']$$

(3)

Note that
$$\mathrm{Tr}[\not{p}\not{p}'] = p_\mu p'_\nu \mathrm{Tr}[\gamma^\mu\gamma^\nu]$$
$$= p_\mu p'_\nu \frac{1}{2}\mathrm{Tr}[\{\gamma^\mu,\gamma^\nu\}]$$
$$= p_\mu p'_\nu \mathrm{Tr}[g^{\mu\nu}\mathbb{1}]$$
$$= 4\, p\cdot p'$$

$$\mathrm{Tr}[\not{p}\gamma_5\not{p}'] = 0$$

$$\Rightarrow \sum_{s,s'} |\bar{u}_s(\vec{p})\, L\, v_{s'}(\vec{p}')|^2 = 2p\cdot p'$$

since $(p+p')^2 = k^2 = M^2$ and $p^2 = p'^2 = 0$

$$\Rightarrow 2p\cdot p' = M^2$$

$$\Rightarrow \sum_{s,s'} |\bar{u}_s(\vec{p})\, L\, v_{s'}(\vec{p}')|^2 = M^2$$

$$\Rightarrow \Gamma_L = \frac{h^2}{2M}\int\frac{d^3p}{(2\pi)^3 2E}\int\frac{d^3p'}{(2\pi)^3 2E'}(2\pi)^4\delta^4(k-p-p')\, M^2 = \frac{h^2}{2M}\cdot\frac{1}{8\pi}\cdot M^2 = \frac{h^2 M}{16\pi}$$

Similarly: $\sum_{s,s'} |\bar{u}_s(\vec{p})\, R\, v_{s'}(\vec{p}')|^2 = \mathrm{Tr}[(\not{p}+m)R(\not{p}'-m)L]$

$$= \frac{1}{4}\mathrm{Tr}[\not{p}(1+\gamma_5)\not{p}'(1-\gamma_5)]$$

$$= \frac{1}{2}\mathrm{Tr}[\not{p}\not{p}' + \not{p}\gamma_5\not{p}']$$

$$= 2p\cdot p' = M^2$$

$$\Rightarrow \Gamma_R = \Gamma_L = \frac{h^2 M}{16\pi}.$$

Part (ii)

(ii) To determine the helicity of anti-fermion we look at (4)

$$\sum_{s,s'} |\bar{u}_s(\vec{p}) R v_{s'}(\vec{p}')| = \sum_{s,s'} [\bar{u}_s(\vec{p}) R v_{s'}(\vec{p}')][\bar{v}_{s'}(\vec{p}') L u_s(\vec{p})]$$

We see that L applies on $u_s(\vec{p})$ while R applies on $v_{s'}(\vec{p}')$, this tells us that the final state is LH particle and RH anti-particle. But the helicity of RH anti-particle is left-handed!

Therefore the helicity of anti-fermion is the same as that of fermion.

Even though the sum is over s and s' (which would be summing over four terms), angular momentum forces two of these to be zero. In other words, given the electron spin, the positron spin is forced to be its opposite.

Part (iii)

To conserve linear momentum, the outgoing electron and positron momenta parallel to the z-axis. Let us say that the electron is traveling along the $+z$ direction.

Along the z axis, the component of orbital angular momentum is zero ($\vec{r} \times \vec{p}$). Therefore, in order for the total angular momentum in this direction to be conserved ($J_z = 0$), the total spin of the electron and positron has to be zero. Therefore, if the electron spin points along the direction of its motion ($+z$ direction), then the positron spin *also* points in the direction of motion of the positron ($-z$ direction). Therefore, it is expected that both positron and the electron are emitted with the same helicity.

Part (iv)

The initial angular momentum is zero. Since total angular momentum is conserved, the total angular momentum in the final state should also be zero.

We are given that the orbital angular momentum for the electron-positron system is $L = 1$. Then, the total spin of the electron-positron system has to be $S = 1$, so that the total $J = 0$.

Since the $S = 1$ state contains the $S_z = 0$ state where both spins point in opposite direction, this is consistent with part(ii) above.

Part (v)

Yes. Different helicities are different final states, and hence the corresponding amplitudes do not interfere. The only interference is when there are multiple amplitudes for the same initial and final states. The two decay widths add to give the total decay width.

8.2.2 Modified interaction

Part (i)

We showed in Homework 6 that $\bar{\psi}\gamma^5\psi$ is odd under parity. Therefore, if ϕ is also odd under parity, then $\mathcal{L}_{int}$ is invariant under parity.

Part (ii)

The spin-sum that arises in this case is

$$\sum_{s,s'} \left| \overline{u_s(\vec{p})}\gamma^5 v_{s'}(\vec{p}')\right|^2 = Tr\left[(\not{p}+m)\gamma^5(\not{p}'-m)\gamma^{5\ddagger}\right] \tag{5}$$

$$\gamma^{5\ddagger} = \gamma^0\gamma^{5\dagger}\gamma^0 = -\gamma^5 \tag{6}$$

Therefore,

$$\sum_{s,s'} \left| \overline{u_s(\vec{p})}\gamma^5 v_{s'}(\vec{p}')\right|^2 = -Tr\left[(\not{p}+m)\gamma^5(\not{p}'-m)\gamma^5\right] \tag{7}$$

$$= Tr\left[(\not{p}+m)(\not{p}'+m)\right] \tag{8}$$

$$= 4p\cdot p' + 4m^2 \tag{9}$$

Now,

$$(p+p')^2 = M^2 \tag{10}$$

$$\Rightarrow 2p\cdot p' = M^2 - p^2 - p'^2 = M^2 - 2m^2 \tag{11}$$

The decay rate can then be written in terms of the phase space ρ as,

$$\Gamma = \frac{|h|^2}{2M}2M^2\rho \tag{12}$$

$$= \frac{|h|^2}{8\pi}M\sqrt{1-\frac{4m^2}{M^2}} \tag{13}$$

where we have used the above formulae from Lahiri and Pal.

Part (iii)

We follow the logic of Problem 5.4.2.

The parity of the initial state is the intrinsic parity of ϕ, $P_\phi = -1$.

The parity of final state $P_{f\bar{f}} = (-1)\times(-1)^L$. Remember that the anti-fermion has an opposite intrinsic parity to the fermion, so the fermion-anti-fermion pair always have a net intrinsic parity of -1.

Therefore, in order for parity to be conserved (which it should be, since $\mathcal{L}_{int}$ was argued to preserve parity above), L should be even.

Now, let us look at angular momentum conservation.

Initial angular momentum $J_\phi = 0$.

Final angular momentum $J_{f\bar{f}} = L_{f\bar{f}} + S_{f\bar{f}}$, where $S_{f\bar{f}} = 0, 1$

Since L has to be even, the final state angular momenta are $L = 0, S = 0$, i.e. the electron-positron pair is in s-wave.

Part (iv)

We can deduce if the final state is in the s-wave or not by looking at the velocity dependence of the transition matrix element squared ($\overline{|\mathcal{M}|^2}$). For s-wave, it is expected to be independent of the velocity ($\sim \beta^0$), and for the p-wave, it is expected to depend on the square of the velocity ($\sim \beta^2$).

In the rest frame of the decaying scalar, the velocity of the electron/positron is given by,

$$\beta = \sqrt{1 - \frac{4m^2}{M^2}} \tag{14}$$

We see that the matrix element for the decay is simply,

$$\overline{|\mathcal{M}|^2} = |h|^2 M \tag{15}$$

and therefore the decay products are in the s-wave.

8.2.3 Most general Yukawa-type interaction

Part (i)

We are given,

$$\begin{aligned} \mathcal{L}_{int} &= -\bar{\psi}(h_S + h_P\gamma_5)\psi\phi & (16)\\ &= -\bar{\psi}(h_L L + h_R R)\psi\phi & (17) \end{aligned}$$

We need to calculate the spin sum as before,

$$\begin{aligned} \sum_{s,s'} \left|\overline{u_s(\vec{p})}(h_L L + h_R R)v_{s'}(\vec{p}\,')\right|^2 &= Tr\left[(\not{p} + m)(h_L L + h_R R)(\not{p}' - m)(h_L L + h_R R)^\ddagger\right] & (18)\\ &= Tr\left[(\not{p} + m)(h_L L + h_R R)(\not{p}' - m)(h_L^* R + h_R^* L)\right] & (19)\\ &= Tr\left[\not{p}(h_L L + h_R R)\not{p}'(h_L^* R + h_R^* L)\right] \\ &\quad - m^2 Tr\left[(h_L L + h_R R)(h_L^* R + h_R^* L)\right] & (20) \end{aligned}$$

Since the trace over a single gamma matrix (with or without additional projectors) is always zero, we don't get a term proportional to a single power of m. Note now,

$$\not{p} L = R \not{p} \tag{21}$$

because of anti-commutation of γ^5 with γ^μ.

$$\sum_{s,s'} \left|\overline{u_s(\vec{p})}(h_L L + h_R R)v_{s'}(\vec{p}\,')\right|^2 = Tr\left[\not{p}\not{p}'(|h_L|^2 R + |h_R|^2 L)\right] - m^2 Tr\left[(h_L h_R^* L + h_R h_L^* R)\right] \tag{22}$$

$$= 2p\cdot p'(|h_L|^2 + |h_R|^2) - 2m^2(h_L h_R^* + h_R h_L^*) \tag{23}$$

where we have carried out the trace as before. The total decay width is now easily given as before,

$$\Gamma = \frac{1}{2M}\left[2p\cdot p'(|h_L|^2+|h_R|^2) - 2m^2(h_L h_R^* + h_R h_L^*)\right]\rho \tag{24}$$

$$= \frac{1}{2M}\left[(M^2-2m^2)(|h_L|^2+|h_R|^2) - 2m^2(h_L h_R^* + h_R h_L^*)\right]\rho \tag{25}$$

$$= \frac{1}{16\pi M}\left[(M^2-2m^2)(|h_L|^2+|h_R|^2) - 2m^2(h_L h_R^* + h_R h_L^*)\right]\sqrt{1-\frac{4m^2}{M^2}} \tag{26}$$

Part (ii)

For $h_L = h_R$,

$$\Gamma = \frac{1}{16\pi M}\left[(M^2-2m^2)(2|h_L|^2) - 2m^2(2|h_L|^2)\right]\sqrt{1-\frac{4m^2}{M^2}} \tag{27}$$

$$= \frac{|h_L|^2 M}{8\pi}\left[1-\frac{4m^2}{M^2}\right]^{\frac{3}{2}} \tag{28}$$

which is the result in Lahiri and Pal.

Similarly, For $h_L = -h_R$,

$$\Gamma = \frac{1}{16\pi M}\left[(M^2-2m^2)(2|h_L|^2) + 2m^2(2|h_L|^2)\right]\sqrt{1-\frac{4m^2}{M^2}} \tag{29}$$

$$= \frac{|h_L|^2 M}{8\pi}\left[1-\frac{4m^2}{M^2}\right]^{\frac{1}{2}} \tag{30}$$

which is the result derived above in Problem 8.2.2(ii).

Part (iii) and (iv)

We can break up the total decay width into three pieces,

$$\Gamma_L = \frac{1}{16\pi M}\left[(M^2-2m^2)|h_L|^2\right]\sqrt{1-\frac{4m^2}{M^2}} \tag{31}$$

$$\Gamma_R = \frac{1}{16\pi M}\left[(M^2-2m^2)|h_R|^2\right]\sqrt{1-\frac{4m^2}{M^2}} \tag{32}$$

$$\Gamma_{int} = \frac{-2m^2}{16\pi M}\left[(h_L h_R^* + h_R h_L^*)\right]\sqrt{1-\frac{4m^2}{M^2}} \tag{33}$$

We see that there is a non-zero interference term between h_L and h_R coupling. Since we are not neglecting masses anymore, the different chirality states are not helicity eigenstates and thus not distinct final states (unlike in the massless limit). Therefore, we get an interference between two chirality amplitudes.

We see that the interference term is proportional to m^2, so as expected, the interference term goes to zero in the massless limit.

8.3 - Measuring chirality

The general form of interaction is

$$\mathcal{L}_{int} = 2\sqrt{2}G_F\, \overline{\psi_{(\nu_e)}}\gamma^\lambda P_e \psi_{(e)}\, \overline{\psi_{(\mu)}}\gamma_\lambda P_\mu \psi_{(\nu_\mu)} \tag{34}$$

P_e and P_μ are projectors which can be chosen to be L, R independently to investigate the helicity structure of this interaction.

8.3.1 R for both electron and muon parts

Part (i) $P_e = P_\mu = R$

In this case,

$$\mathcal{M}_{fi} = \frac{G_F}{\sqrt{2}}\bar{u}_{(\nu_e)}(k')\gamma^\lambda(1+\gamma^5)u_{(e)}(p)\; \bar{u}_{(\mu)}(p')\gamma_\lambda(1+\gamma^5)u_{(\nu_\mu)}(k) \tag{35}$$

Since we have both left-handed and right-handed neutrinos, an initially unpolarized beam will contain both these components. Therefore, we should add over both helicities and then divide by two for the initial state spin-sum.

In this particular case, since we use a R projector, only the right-handed helicities will contribute when we sum over the spins. Squaring and summing over final states and averaging over initial states,

$$\begin{aligned}
\overline{|\mathcal{M}_{fi}|^2} &= \frac{1}{4}\sum_{spin}|\mathcal{M}_{fi}|^2 && (36)\\
&= \frac{G_F^2}{8}Tr\left[(\not p + m_e)\gamma^\rho(1+\gamma^5)\not k'\gamma^\lambda(1+\gamma^5)\right] Tr\left[\not k\gamma_\rho(1+\gamma^5)(\not p' + m_\mu)\gamma_\lambda(1+\gamma^5)\right] && (37)\\
&= \frac{G_F^2}{2}Tr\left[(\not p + m_e)\gamma^\rho(1+\gamma^5)\not k'\gamma^\lambda\right] Tr\left[\not k\gamma_\rho(1+\gamma^5)(\not p' + m_\mu)\gamma_\lambda\right] && (38)\\
&= \frac{G_F^2}{2}Tr\left[\not p\gamma^\rho(1+\gamma^5)\not k'\gamma^\lambda\right] Tr\left[\not k\gamma_\rho(1+\gamma^5)\not p'\gamma_\lambda\right] && (39)\\
&= \frac{G_F^2}{2}p_\alpha k'_\beta k^\sigma p'^\delta Tr\left[\gamma^\alpha\gamma^\rho(1+\gamma^5)\gamma^\beta\gamma^\lambda\right] Tr\left[\gamma_\sigma\gamma_\rho(1+\gamma^5)\gamma_\delta\gamma_\lambda\right] && (40)\\
&= 8G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta\left[g^{\alpha\rho}g^{\beta\lambda} - g^{\alpha\beta}g^{\rho\lambda} + g^{\alpha\lambda}g^{\beta\rho} + i\epsilon^{\beta\lambda\alpha\rho}\right]\left[g_{\sigma\rho}g_{\delta\lambda} - g_{\sigma\delta}g_{\rho\lambda} + g_{\sigma\lambda}g_{\delta\rho} + i\epsilon_{\delta\lambda\sigma\rho}\right] && (41)\\
&= 8G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta\left[2g^\alpha_\sigma g^\beta_\delta + 2g^\alpha_\delta g^\beta_\sigma - \epsilon^{\alpha\beta\lambda\rho}\epsilon_{\sigma\delta\lambda\rho}\right] && (42)\\
&= 8G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta\left[2g^\alpha_\sigma g^\beta_\delta + 2g^\alpha_\delta g^\beta_\sigma + 2(g^\alpha_\sigma g^\beta_\delta - g^\alpha_\delta g^\beta_\sigma)\right] && (43)\\
&= 32G_F^2\; p\cdot k\; p'\cdot k' && (44)
\end{aligned}$$

This is the same as the result obtained in Lahiri and Pal. The kinematics are obviously identical (Section 7.5.1 in L& P), so the differential cross-section is the same as obtained in Equation 7.130 (modulo the additional factor of 2 discussed above).

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2}{8\pi^2}\frac{(s-m_\mu^2)^2}{s} \tag{45}$$

8.3.2 R for electron part and L for muon part

Part (i) $P_e = R, P_\mu = L$

Now, in this case,

$$\mathcal{M}_{fi} = \frac{G_F}{\sqrt{2}} \bar{u}_{(\nu_e)}(k')\gamma^\lambda(1+\gamma^5)u_{(e)}(p)\ \bar{u}_{(\mu)}(p')\gamma_\lambda(1-\gamma^5)u_{(\nu_\mu)}(k) \tag{46}$$

The calculation proceeds in the same way. Squaring and summing over final states and averaging over initial states,

$$\overline{|\mathcal{M}_{fi}|^2} = \frac{1}{4}\sum_{spin}|\mathcal{M}_{fi}|^2 \tag{47}$$

$$= \frac{G_F^2}{8} Tr\left[(\not{p}+m_e)\gamma^\rho(1+\gamma^5)\not{k}'\gamma^\lambda(1+\gamma^5)\right] Tr\left[\not{k}\gamma_\rho(1-\gamma^5)(\not{p}'+m_\mu)\gamma_\lambda(1-\gamma^5)\right] \tag{48}$$

$$= \frac{G_F^2}{2} p_\alpha k'_\beta k^\sigma p'^\delta Tr\left[\gamma^\alpha\gamma^\rho(1+\gamma^5)\gamma^\beta\gamma^\lambda\right] Tr\left[\gamma_\sigma\gamma_\rho(1-\gamma^5)\gamma_\delta\gamma_\lambda\right] \tag{49}$$

$$= 8G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta \left[g^{\alpha\rho}g^{\beta\lambda} - g^{\alpha\beta}g^{\rho\lambda} + g^{\alpha\lambda}g^{\beta\rho} + i\epsilon^{\beta\lambda\alpha\rho}\right]\left[g_{\sigma\rho}g_{\delta\lambda} - g_{\sigma\delta}g_{\rho\lambda} + g_{\sigma\lambda}g_{\delta\rho} - i\epsilon_{\delta\lambda\sigma\rho}\right] \tag{50}$$

$$= 8G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta \left[2g^\alpha_\sigma g^\beta_\delta + 2g^\alpha_\delta g^\beta_\sigma + \epsilon^{\alpha\beta\lambda\rho}\epsilon_{\sigma\delta\lambda\rho}\right] \tag{51}$$

$$= 8G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta \left[2g^\alpha_\sigma g^\beta_\delta + 2g^\alpha_\delta g^\beta_\sigma - 2(g^\alpha_\sigma g^\beta_\delta - g^\alpha_\delta g^\beta_\sigma)\right] \tag{52}$$

$$= 32G_F^2\ p\cdot p'\ k\cdot k' \tag{53}$$

The kinematic dependence is different from the case above. We can use the expressions derived in the lectures (and the notes on neutrino-electron scattering),

$$p\cdot p' = \frac{(s+m_e^2)(s+m_\mu^2) - (s-m_e^2)(s-m_\mu^2)\cos\theta}{4s} \tag{54}$$

$$k\cdot k' = \frac{(s-m_e^2)(s-m_\mu^2)(1-\cos\theta)}{4s} \tag{55}$$

The differential cross-section is,

$$\frac{d\sigma}{d\Omega} = \frac{32G_F^2}{64\pi^2 s}\left[\frac{s-m_\mu^2}{s-m_e^2}\right]\left[\frac{(s+m_e^2)(s+m_\mu^2) - (s-m_e^2)(s-m_\mu^2)\cos\theta}{4s}\right]\left[\frac{(s-m_e^2)(s-m_\mu^2)(1-\cos\theta)}{4s}\right] \tag{56}$$

$$= \frac{G_F^2(s-m_e^2)(s-m_\mu^2)^3}{32\pi^2 s^3}(1-\cos\theta)\left[\frac{(s+m_e^2)(s+m_\mu^2)}{(s-m_e^2)(s-m_\mu^2)} - \cos\theta\right] \tag{57}$$

Part (ii)

When the angle between the muon and electron is zero, $\theta = 0$, the differential cross-section vanishes, even without m_e or $m_\mu = 0$. When $m_e = m_\mu = 0$, the second term additionally vanishes.

The Dirac structure for the muon part (with the L insertion)

$$\begin{aligned}
\bar{u}_{(\mu)}(p')\gamma_\lambda L u_{(\nu_\mu)}(k) &= \bar{u}_{(\mu)}(p')\gamma_\lambda L L u_{(\nu_\mu)}(k) && (58)\\
&= \bar{u}_{(\mu)}(p') R \gamma_\lambda L u_{(\nu_\mu)}(k) && (59)\\
&= u^\dagger_{(\mu)}(p')\gamma^0 R \gamma_\lambda L u_{(\nu_\mu)}(k) && (60)\\
&= u^\dagger_{(\mu)}(p') L \gamma^0 \gamma_\lambda L u_{(\nu_\mu)}(k) && (61)\\
&= (Lu)^\dagger_{(\mu)}(p')\gamma^0 \gamma_\lambda L u_{(\nu_\mu)}(k) && (62)\\
&= \overline{Lu}_{(\mu)}(p')\gamma_\lambda L u_{(\nu_\mu)}(k) && (63)
\end{aligned}$$

Therefore, this particular projector insertion creates a left-handed muon in the final state and annihilates a left-handed ν_μ in the initial state. Similarly, we can show that the electron and ν_e are right-handed particles.

Now we can understand the vanishing of differential cross-section in terms of angular momentum. Let us assume that the incoming electron was moving along the z-axis with its momentum in the $+z$ direction. Along the beam direction (which is both incoming and outgoing particles directions for $\theta = 0$), the orbital angular momentum is zero. The initial state electron is right-handed and the muon neutrino is left-handed, so both their spins point in the $+z$ direction (along the direction of motion for a left handed electron, opposite to the direction of motion for the right handed muon neutrino). For $\theta = 0$, the outgoing muon moves in the $+z$ direction and since it is a left-handed fermion, its spin points in $-z$ direction. Similarly the spin of the outgoing electron neutrino also points in the $-z$ direction. Since angular momentum in the initial and final states are forced to be different at $\theta = 0$, the differential cross-section at this angle vanishes.

8.3.3 No L, R

Part (i) $P_e = P_\mu = 1$

Now there are no projectors in the amplitude.

$$\mathcal{M}_{fi} = \frac{4G_F}{\sqrt{2}} \bar{u}_{(\nu_e)}(k')\gamma^\lambda u_{(e)}(p)\ \bar{u}_{(\mu)}(p')\gamma_\lambda u_{(\nu_\mu)}(k) \tag{64}$$

The calculation proceeds in the same way. Squaring and summing over final states and averaging over initial states,

$$\begin{aligned}
\overline{|\mathcal{M}_{fi}|^2} &= \frac{1}{4}\sum_{spin} |\mathcal{M}_{fi}|^2 && (65)\\
&= 2G_F^2 Tr\left[(\not{p} + m_e)\gamma^\rho \not{k}'\gamma^\lambda\right] Tr\left[\not{k}\gamma_\rho(\not{p}' + m_\mu)\gamma_\lambda\right] && (66)\\
&= 32G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta \left[g^{\alpha\rho}g^{\beta\lambda} - g^{\alpha\beta}g^{\rho\lambda} + g^{\alpha\lambda}g^{\beta\rho}\right]\left[g_{\sigma\rho}g_{\delta\lambda} - g_{\sigma\delta}g_{\rho\lambda} + g_{\sigma\lambda}g_{\delta\rho}\right] && (67)\\
&= 32G_F^2 p_\alpha k'_\beta k^\sigma p'^\delta \left[2g^\alpha_\sigma g^\beta_\delta + 2g^\alpha_\delta g^\beta_\sigma\right] && (68)\\
&= 64G_F^2 \left(p\cdot p'\ k\cdot k' + p\cdot k\ p'\cdot k'\right) && (69)
\end{aligned}$$

The differential cross-section is,

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2(s-m_e^2)(s-m_\mu^2)^3}{16\pi^2 s^3}(1-\cos\theta)\left[\frac{(s+m_e^2)(s+m_\mu^2)}{(s-m_e^2)(s-m_\mu^2)} - \cos\theta\right] + \frac{G_F^2}{4\pi^2}\frac{(s-m_\mu^2)^2}{s} \quad (70)$$

Part (ii)

We see that the differential cross-section is twice the sum of the results we got in part 8.2.1 and 8.2.2. This is easily understood to be the sum of contributions from the LL, LR, RL and RR projectors.

In general we would expect there to be no interference in the helicity amplitudes in the massless limit (since chirality = helicity in that limit), as discussed in the decay example. However, in this case it turns out to be true even when the electrons and muons are massive (the neutrinos are still assumed to be massless though). This can be understood as follows.

We can always move around the chirality projector P_e in the amplitude so that the projectors act on the neutrino spinors. Since the neutrino is massless, the chirality projector is also the helicity projector. Therefore, choosing different values of P_e, P_μ can be thought as choosing different helicities for the neutrinos in the initial and final states. Different neutrino helicities are distinct final states, and hence these helicity amplitudes do not interfere with one another (even though chirality $\neq$ helicity for e, μ, i.e. the final states for e, μ are not distinct helicity eigenstates).