Homework 7 Solutions

7.1 - Drawing Feynman diagrams

We choose a convention where time is flowing towards the right. So incoming states (external lines) appear on the left, and outgoing external lines appear on the right.

7.1.1 - Scalar-fermion scattering

problem 7.1.1

scattering process $Be^- \to Be^-$

initial state $|B(k), e^-(k')\rangle$

final state $|B(p), e^-(p')\rangle$

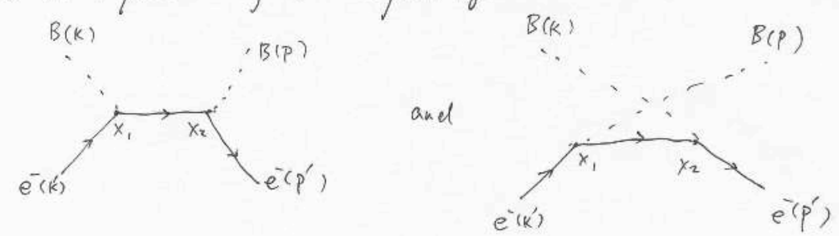
Interaction Hamiltonian: $\mathcal{H}_I = h :\bar{\psi}\psi\phi:$

Obviously, we need at least two $\mathcal{H}_I$ to annihilate the initial state and create the final state:

The matrix element is given by:

$$S^{(2)}_{fi} = \frac{(-ih)^2}{2!}\int d^4x_1\, d^4x_2 \,\langle B(p), e^-(p')|\, \mathcal{T}\left[:(\bar{\psi}\psi\phi):_{x_1} :(\bar{\psi}\psi\phi):_{x_2}\right] |B(k), e^-(k')\rangle$$

Then the Feynman diagrams are given by

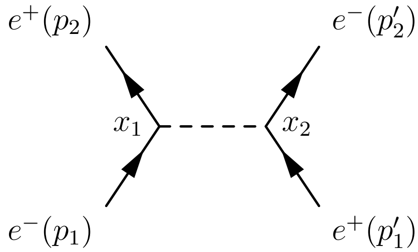


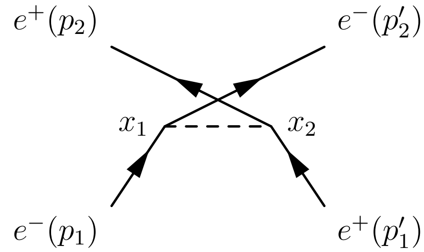
7.1.2 - Fermion-antifermion scattering

As before, we need two copies of the interaction Hamiltonian,

$$S_{fi} = \frac{(-ih)^2}{2!}\int d^4x_1\, d^4x_2 \left\langle e^+(p_1')e^-(p_2')\right| \mathcal{T}(:\bar{\psi}\psi\phi:)_{x_1}(:\bar{\psi}\psi\phi:)_{x_2} \left|e^+(p_1)e^-(p_2)\right\rangle \tag{1}$$

Therefore, the Feynman diagrams contributing to this process look like the following,



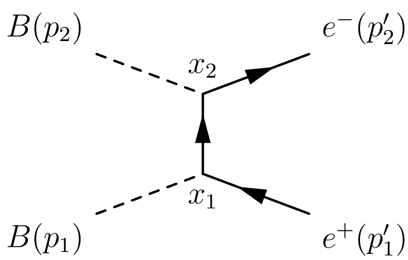


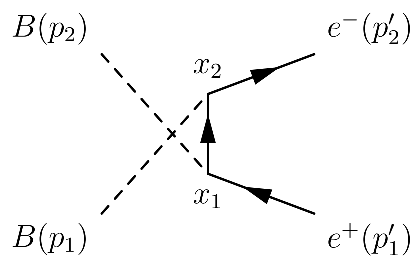
As was pointed out, the diagrams look similar to the scalar-fermion scattering case.

7.1.3 - Annihilation

Part (i)

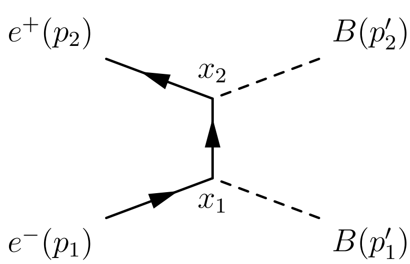
The scalar annihilation diagram is obtained by "rotating" the diagrams in part (7.1.1) 90 degrees anti-clockwise. A useful mnemonic is that when we switch an incoming particle (the electron) to an outgoing particle, we make it into an anti-particle. We can see this from the diagram. If there is an arrow on the fermion line opposite to its momentum line (e.g. incoming fermion arrow on an outgoing particle), that fermion line corresponds to the anti-fermion.

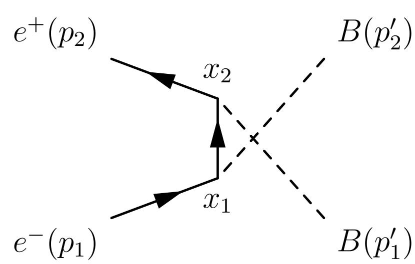




Part (ii)

To get the diagrams for $e^+e^- \to BB$, one can flip the diagrams above. Keeping track of fermion line arrows tell us which external line corresponds to fermions and which to anti-fermions.





We have also changed the momenta labels to keep with the convention that unprimed variables denote incoming particle momenta, and primed variables denote outgoing particle momenta.

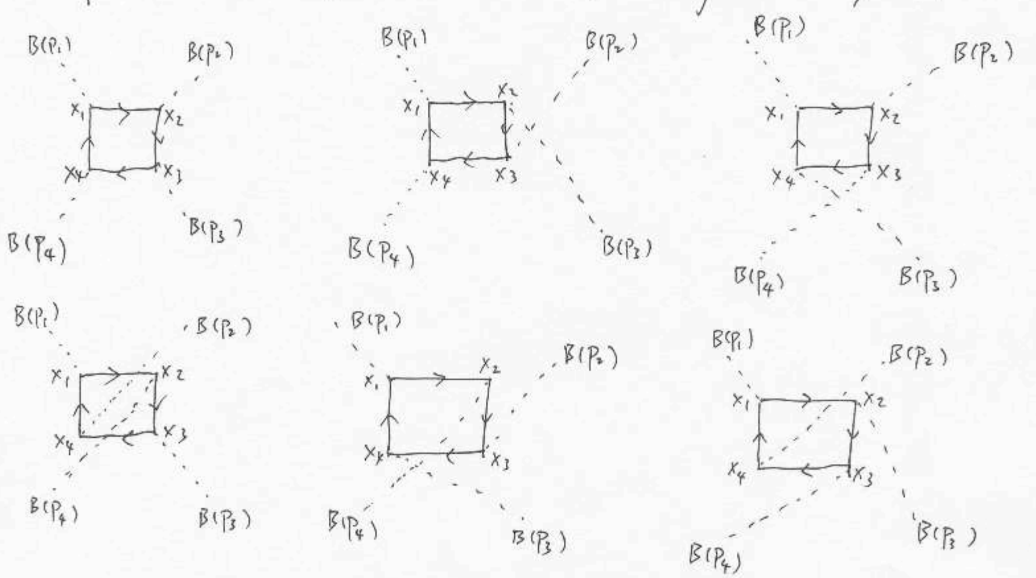
7.1.4 - Scalar-scalar scattering

problem 7.1.2 / Ex 6.1 of L&P

We consider only connected diagrams since only them contribute to scattering amplitude.

Obviously, we need a fermion loop. Let's denote the initial and final state as $|B(P_1)\,B(P_2)\rangle$ and $|B(P_3)\,B(P_4)\rangle$; and we mark the 4 interaction points as x_1, x_2, x_3, x_4.

There are $4! = 24$ ways to connect the external legs to the 4 interaction points. However, a rotation of the interaction points give the same diagram. Therefore, we have $\frac{24}{4} = 6$ distinct Feynman diagrams. They are:



7.1.5 - All possible $2 \to 2$ processes

All possible $2 \to 2$ processes involving the (anti)fermions are,

$$e^+e^+ \to e^+e^+ \tag{2}$$
$$e^+e^- \to e^+e^- \tag{3}$$
$$e^-e^- \to e^-e^- \tag{4}$$

The final state is fixed given the initial state by charge conservation. Processes involving two scalar external states,

$$e^+B \to e^+B \tag{5}$$
$$e^-B \to e^-B \tag{6}$$
$$BB \to e^-e^+ \tag{7}$$
$$e^-e^+ \to BB \tag{8}$$

And finally, we have the scalar-scalar scattering,

$$BB \to BB \tag{9}$$

We have 6 distinct initial states. Four of those have net charge, and hence one fixed final states. Two have no net charge, and give two possible final states each. So the total number of processes is 8. These are all the $2 \to 2$ processes allowed in this theory.

7.2 - Contraction of field operators

$$\phi_+(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} e^{-ip\cdot x} a(p)$$

$$|B(\vec{k})\rangle = \sqrt{\frac{(2\pi)^3}{V}}\, a^\dagger(k)\,|0\rangle$$

$$\Rightarrow \phi_+(x)\,|B(\vec{k})\rangle = \int \frac{d^3p}{\sqrt{(2\pi)^3\cdot 2E_p}} e^{-ip\cdot x}\sqrt{\frac{(2\pi)^3}{V}}\cdot a(p)\,a^\dagger(k)\,|0\rangle$$

note that $[a(p), a^\dagger(k)] = \delta^3(\vec{p}-\vec{k})$ and $a(p)|0\rangle = 0$

$$\Rightarrow \phi_+(x)\,|B(\vec{k})\rangle = \int \frac{d^3p}{\sqrt{2E_p}} e^{-ip\cdot x}\sqrt{\frac{1}{V}}\,\delta^3(\vec{p}-\vec{k})\,|0\rangle = \sqrt{\frac{1}{2E_k V}}\cdot e^{-ik\cdot x}\,|0\rangle.$$

(3)

$$\psi_+(x) = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} \sum_{s'=1,2} f_{s'}(p')\, u_{s'}(p')\, e^{-ip'\cdot x}$$

$$|e^-(p,s)\rangle = \sqrt{\frac{(2\pi)^3}{V}}\, f_s^\dagger(p)\,|0\rangle$$

$$\Rightarrow \psi_+(x)\,|e^-(p,s)\rangle = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} \sum_{s'=1,2} f_{s'}(p')\,f_s^\dagger(p)\,|0\rangle\cdot e^{-ip'\cdot x}\sqrt{\frac{(2\pi)^3}{V}}\cdot u_{s'}(p')$$

$$= \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} \sum_{s'=1,2} \delta_{s's}\,\delta^3(\vec{p}'-\vec{p})\,|0\rangle\, e^{-ip'\cdot x}\sqrt{\frac{(2\pi)^3}{V}}\cdot u_{s'}(p')$$

$$= \sqrt{\frac{1}{2E_p V}}\cdot e^{-ip\cdot x}\, u_s(p)\,|0\rangle$$

$$\bar{\psi}_+(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1,2} \hat{f}_s(p)\,\bar{v}_s(p)\,e^{-ip\cdot x}$$

$$|e^+(p',s')\rangle = \sqrt{\frac{(2\pi)^3}{V}}\, \hat{f}^\dagger_{s'}(\vec{p}')\,|0\rangle$$

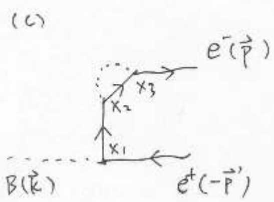
$$\bar{\psi}_+(x)\,|e^+(p',s')\rangle = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1,2} \hat{f}_s(p)\,\bar{v}_s(p)\,e^{-ip\cdot x}\sqrt{\frac{(2\pi)^3}{V}}\,\hat{f}^\dagger_{s'}(p')\,|0\rangle$$

$$= \frac{1}{\sqrt{2E_{p'}V}}\cdot \bar{v}_{s'}(p')\,e^{-ip'\cdot x}\,|0\rangle$$

$\square$

7.3 - Expression for S-matrix element

(c)



$$S_{fi}^{(c)} = (-ih)^3 \int d^4x_1\, d^4x_2\, d^4x_3 \;\; i\Delta_F(x_3-x_2)\, iS_{F\beta\alpha}(x_3-x_2)$$
$$\times\, iS_{F\alpha\gamma}(x_2-x_1) \left\langle e^-(p)\, e^+(p') \right| \bar{\psi}_-^{\beta}(x_3)\, \psi_-^{\gamma}(x_1)\, \phi_+(x_1) \left| B(\vec{k}) \right\rangle$$

4

$$S_{fi}^{(c)} = (-ih)^3 \int d^4x_1\, d^4x_2\, d^4x_3 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q_3}{(2\pi)^4}$$
$$\times\, i\Delta_F(q_1)\, e^{-iq_1\cdot(x_3-x_2)}\; iS_{F\beta\alpha}(q_2)\, e^{-iq_2\cdot(x_3-x_2)}\; iS_{F\alpha\gamma}(q_3)\, e^{-iq_3\cdot(x_2-x_1)}$$
$$\times \left[\frac{e^{-ik\cdot x_1}}{\sqrt{2\omega_k V}} \cdot \frac{\bar{u}_s^{\beta}(p)\, e^{ip\cdot x_3}}{\sqrt{2E_p V}} \cdot \frac{v_{s'}^{\gamma}(p')\, e^{ip'\cdot x_1}}{\sqrt{2E_{p'} V}} \right]$$

The $\int d^4x_1\, d^4x_2\, d^4x_3$ integration gives delta functions:

$$S_{fi}^{(c)} = (-ih)^3 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4} \cdot (2\pi)^{12}\, \delta^4(q_3 - k + p')\, \delta^4(q_1 + q_2 - q_3)$$
$$\times\, \delta^4(-q_1 - q_2 + p)\; i\Delta_F(q_1)\; \cancel{iS_{F\beta\alpha}(q_2)\, iS_{F\alpha\gamma}(q_3)}$$
$$\times \left[\bar{u}_s(\vec{p})\, iS_F(q_2)\, iS_F(q_3)\, v_{s'}(\vec{p}') \right] \frac{1}{\sqrt{2\omega_k V}} \cdot \frac{1}{\sqrt{2E_p V}} \cdot \frac{1}{\sqrt{2E_{p'} V}}$$
$$= (-ih)^3 \int \frac{d^4q}{(2\pi)^4} \cdot (2\pi)^4 \delta^4(k-p-p')\; i\Delta_F(q) \left[\bar{u}_s(\vec{p})\, iS_F(p-q)\, iS_F(k-p')\, v_{s'}(\vec{p}') \right]$$
$$\times \left[\frac{1}{\sqrt{2\omega_k V}} \frac{1}{\sqrt{2E_p V}} \frac{1}{\sqrt{2E_{p'} V}} \right]$$
$$= (-ih)^3 (2\pi)^4 \delta^4(k-p-p') \int \frac{d^4q}{(2\pi)^4} \left(\frac{i}{q^2 - m_B^2 + i\varepsilon} \right) \left[\bar{u}_s(\vec{p}) \frac{i}{\not{p} - \not{q} - m_e + i\varepsilon} \cdot \frac{i}{\not{p} - m_e + i\varepsilon}\, v_{s'}(\vec{p}') \right]$$
$$\times \left[\frac{1}{\sqrt{2\omega_k V}} \cdot \frac{1}{\sqrt{2E_p V}} \cdot \frac{1}{\sqrt{2E_{p'} V}} \right)$$

(d)

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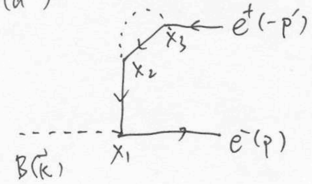

$$S_{fi}^{(d)} = (-i\hbar)^3 \int d^4x_1 d^4x_2 d^4x_3 \; i\Delta_F(x_2-x_3)$$

$$\times\, iS_{F_{\beta\alpha}}(x_2-x_3)\; iS_{F_{\gamma\beta}}(x_1-x_2)$$

$$\times \left\langle e^-(p)\, e^+(p') \right| \bar{\psi}_-^{\gamma}(x_1)\, \psi_-^{\alpha}(x_3)\, \phi_+(x_1) \left| B(\vec{k}) \right\rangle$$

$$= (-i\hbar)^3 \int d^4x_1 d^4x_2 d^4x_3 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4q_3}{(2\pi)^4}\; i\Delta_F(q_1)\, e^{-iq_1\cdot(x_2-x_3)}$$

$$\times\, iS_{F_{\beta\alpha}}(q_2)\, e^{-iq_2\cdot(x_2-x_3)}\; iS_{F_{\gamma\beta}}(q_3)\, e^{-iq_3\cdot(x_1-x_2)}$$

$$\times \left[\frac{e^{-ik\cdot x_1}}{\sqrt{2\omega_k V}} \cdot \frac{\bar{u}_s^{\gamma}(p)\, e^{ip\cdot x_1}}{\sqrt{2E_p V}} \cdot \frac{v_{s'}^{\beta}(p')\, e^{ip'\cdot x_3}}{\sqrt{2E_{p'} V}} \right]$$

$$= (-i\hbar)^3 \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \cdot \frac{d^4q_3}{(2\pi)^3} \cdot (2\pi)^{12} \cdot \delta^4(q_3+k-p)\, \delta^4(q_1+q_2-q_3)\, \delta^4(-q_1-q_2-p')$$

$$\times\, i\Delta_F(q_1) \times \left[\bar{u}_s(p)\, iS_F(q_3)\, iS_F(q_2)\, v_{s'}(p') \right] \frac{1}{\sqrt{2\omega_k V}} \cdot \frac{1}{\sqrt{2E_p V}} \cdot \frac{1}{\sqrt{2E_{p'} V}}$$

$$= (-i\hbar)^3 \cdot (2\pi)^4 \delta^4(k-p-p') \cdot \int \frac{d^4q}{(2\pi)^4}\, i\Delta_F(q) \left[\bar{u}_s(p)\, iS_F(p-k)\, iS_F(-p'-q)\, v_{s'}(p') \right]$$

$$\times \left[\frac{1}{\sqrt{2\omega_k V}} \cdot \frac{1}{\sqrt{2E_p V}} \cdot \frac{1}{\sqrt{2E_{p'} V}} \right]$$

$$= (-i\hbar)^3 (2\pi)^4 \delta^4(k-p-p') \int \frac{d^4q}{(2\pi)^4} \left(\frac{i}{q^2-m_B^2+i\varepsilon} \right) \left[\bar{u}_s(p) \frac{i}{\not{p}-\not{k}-m_e+i\varepsilon} \cdot \frac{i}{-\not{p}'-\not{q}-m_e+i\varepsilon}\, v_{s'}(p') \right]$$

$$\times \left[\frac{1}{\sqrt{2\omega_k V}} \cdot \frac{1}{\sqrt{2E_p V}} \cdot \frac{1}{\sqrt{2E_{p'} V}} \right].$$

□

7.4 - Identical scalar particles

Calculate $a(\vec{k})\, a(\vec{k}')\, a^{+}(\vec{p}_2)\, a^{+}(\vec{p}_1)\, |0\rangle$

use the commutation relation

$$[a(\vec{k}),\, a^{+}(\vec{p})] = \delta^3(\vec{k}-\vec{p})$$

$$\Rightarrow a(\vec{k})\, a(\vec{k}')\, a^{+}(\vec{p}_2)\, a^{+}(\vec{p}_1)\, |0\rangle$$

$$= \left[a(\vec{k})\, \delta^3(\vec{k}'-\vec{p}_2)\, a^{+}(\vec{p}_1) + a(\vec{k})\, a^{+}(\vec{p}_2)\, a(\vec{k}')\, a^{+}(\vec{p}_1)\right]|0\rangle$$

$$= \left[\delta^3(\vec{k}'-\vec{p}_2)\, \delta^3(\vec{k}-\vec{p}_1) + a(\vec{k})\, a^{+}(\vec{p}_2)\, \delta^3(\vec{k}'-\vec{p}_1)\right]|0\rangle$$

$$= \left[\delta^3(\vec{k}'-\vec{p}_2)\, \delta^3(\vec{k}-\vec{p}_1) + \delta^3(\vec{k}-\vec{p}_2)\, \delta^3(\vec{k}'-\vec{p}_1)\right]|0\rangle$$

where we used $a|0\rangle = 0$.

We can see that there is no negative sign between two Feynman diagrams. This result conforms with spin-statistics theorem since exchange of two scalar does not introduce minus sign.