Homework 6 Solutions

6.1 - Restriction on interaction Lagrangian

6.1.1 - Hermiticity

(i) $\mathcal{L}_{int} = h\phi\bar{\psi}\psi$; ϕ is real

$$\Rightarrow \mathcal{L}_{int}^{\dagger} = h^*\phi\,(\psi^{\dagger}\gamma^0\psi)^{\dagger}$$
$$= h^*\phi\,\psi^{\dagger}\gamma^{0\dagger}\psi$$

recall that $\gamma^{0\dagger} = \gamma^0$

$$\Rightarrow \mathcal{L}_{int}^{\dagger} = h^*\phi\,\psi^{\dagger}\gamma^0\psi = h^*\phi\bar{\psi}\psi$$

in order for $\mathcal{L}_{int}$ to be hermitian, we need $h = h^*$, i.e., h is real.

(ii) $\mathcal{L}_{int} = h\phi\bar{\psi}\gamma_5\psi$

$$\Rightarrow \mathcal{L}_{int}^{\dagger} = h^*\phi\,(\psi^{\dagger}\gamma^0\gamma_5\psi)^{\dagger}$$
$$= h^*\phi\,\psi^{\dagger}\gamma_5^{\dagger}\gamma_0^{\dagger}\psi$$

recall $\gamma_5^{\dagger} = \gamma_5$, $\gamma_0^{\dagger} = \gamma_0$ and $\{\gamma_5, \gamma_0\} = 0$

$$\Rightarrow \mathcal{L}_{int}^{\dagger} = -h^*\phi\,\psi^{\dagger}\gamma_0\gamma_5\psi = -h^*\phi\bar{\psi}\gamma_5\psi$$

For $\mathcal{L}_{int}$ to be hermitian, we need $h = -h^*$, i.e., h has to be purely imaginary

(iii) $\mathcal{L}_{int} = h\phi\bar{\psi}L\psi = \frac{1}{2}h\phi\bar{\psi}\psi - \frac{1}{2}h\phi\bar{\psi}\gamma_5\psi$

From previous analysis, for the first term to be hermitian, h should be real; for the second term to be hermitian, h should be imaginary. Therefore $\mathcal{L}_{int}$ cannot be hermitian.

To make it hermitian, we just need to add the hermitian conjugate of $\mathcal{L}_{int}$

$$\mathcal{L}_{int}' = \mathcal{L}_{int} + \mathcal{L}_{int}^{\dagger} = h\phi\bar{\psi}\left(\frac{1-\gamma_5}{2}\right)\psi + h^*\phi\bar{\psi}\left(\frac{1+\gamma_5}{2}\right)\psi$$
$$= h\phi\bar{\psi}L\psi + h^*\phi\bar{\psi}R\psi$$

where $R \equiv \frac{1+\gamma_5}{2}$

Therefore, we need to add $h^*\phi\bar{\psi}R\psi$ to the Lagrangian to make it hermitian.

6.1.2 - Lorentz invariance

We borrow the following results from homework 4. Under a Lorentz transformation, the bilinears transform as follows,

$$\psi \to \Lambda_{\frac{1}{2}}\psi \tag{1}$$
$$\bar{\psi} \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1} \tag{2}$$

and the fact that $\Lambda_{\frac{1}{2}}$ commutes with γ^5. This can be derived using the fact that γ^5 anti-commutes with all γ^μ.

Part (i)

It is now easy to calculate the transformation property,

$$\bar{\psi}\gamma^5\psi \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\gamma^5\Lambda_{\frac{1}{2}}\psi \tag{3}$$
$$= \bar{\psi}\gamma^5\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\psi \tag{4}$$
$$= \bar{\psi}\gamma^5\psi \tag{5}$$

Thus, it transforms the same way as a scalar under continuous Lorentz transformation. As shown elsewhere in this homework, this operator is odd under parity, so it is a "pseudo-scalar" operator.

Part (ii)

Given

$$\mathcal{L}_{int} = (\bar{\psi}\gamma^5\psi)(\bar{\psi}\gamma^5\psi) \tag{6}$$

Since each of the terms is separately Lorentz (pseudo-scalar), the interaction term is Lorentz invariant as well.

Part (iii)

Repeating the calculation above,

$$\Lambda^\nu_\mu\bar{\psi}\gamma^\mu\gamma^5\psi \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda^\nu_\mu\gamma^5\Lambda_{\frac{1}{2}}\psi \tag{7}$$
$$= \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda^\nu_\mu\gamma^\mu\Lambda_{\frac{1}{2}}\gamma^5\psi \tag{8}$$
$$= \bar{\psi}\gamma^\nu\gamma^5\psi \tag{9}$$

where we have used $\Lambda_{\frac{1}{2}}^{-1}\Lambda^\nu_\mu\gamma^\mu\Lambda_{\frac{1}{2}} = \gamma^\nu$, proved in an earlier homework.

Thus, this spinor transforms like a vector under continuous Lorentz transformations. Again, this is seen to behave opposite to a vector under parity, so it is an "axial-vector" or a "pseudo-vector".

Part (iv)

Given,

$$\mathcal{L}_{int} = (\bar{\psi}\gamma^\mu\gamma^5\psi)(\bar{\psi}\gamma_\mu\gamma^5\psi) \tag{10}$$

Since each of the terms above transforms honestly with the Lorentz index, the interaction made from contracted Lorentz indices is a Lorentz invariant.

6.1.3 - Renormalizability

The basis of estimating mass dimensions is the fact that the action is dimensionless in natural units ($\hbar = c = 1$). This can be seen from the time-evolution operator ($\sim e^{i\int dtH}$) or from the path-integral ($\sim \int e^{iS}$).

Part (i)

The action for the kinetic term is

$$S = \int d^4x \;\bar{\psi}\; \partial\!\!\!/ \;\psi \tag{11}$$

The γ-matrices and the Lorentz indices are unimportant for the dimension calculation. We get 4 inverse powers of mass from the volume integration, one power of mass from the derivative. Therefore,

$$2[\psi] - 4 + 1 = 0 \tag{12}$$

$$\Rightarrow [\psi] = \frac{3}{2} \tag{13}$$

Part (ii)

The action for the mass term,

$$S = \int d^4x \; m\bar{\psi}\psi \tag{14}$$

Thus,

$$2[\psi] - 4 + [m] = 0 \tag{15}$$

$$\Rightarrow [m] = 1 \tag{16}$$

Part (iii)

We need the dimension of the scalar field. The action for the Yukawa term (with the kinetic term for the scalar included),

$$S = \int d^4x \; \lambda\phi\bar{\psi}\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{17}$$

Thus,

$$2[\phi] - 4 + 2 = 0 \tag{18}$$
$$\Rightarrow [\phi] = 1 \tag{19}$$
$$[\phi\bar{\psi}\psi] - 4 + [\lambda] = 0 \tag{20}$$
$$\Rightarrow [\lambda] = 0 \tag{21}$$

Part (iv)

For the four-fermion interaction

$$S = \int d^4x \;\; \kappa(\bar{\psi}\psi)(\bar{\psi}\psi) \tag{22}$$

$$4[\psi] - 4 + [\kappa] = 0 \tag{23}$$
$$\Rightarrow [\kappa] = -2 \tag{24}$$

Since the co-efficient has negative mass-dimensions, it turns out that this interaction is non-renormalizable.

6.1.4 - Parity invariance

Part (i)

Under parity,

$$\psi \rightarrow \psi_p = \eta_p \gamma^0 \psi \tag{25}$$

where $|\eta_p|^2 = 1$. Therefore,

$$\bar{\psi}\gamma^5\psi \rightarrow \psi_p^\dagger \gamma^0 \gamma^5 \psi_p \tag{26}$$
$$= |\eta_p|^2 \psi^\dagger \gamma^{0\dagger} \gamma^0 \gamma^5 \gamma^0 \psi \tag{27}$$
$$= \psi^\dagger \gamma^5 \gamma^0 \psi \tag{28}$$
$$= -\psi^\dagger \gamma^0 \gamma^5 \psi \tag{29}$$
$$= -\bar{\psi}\gamma^5\psi \tag{30}$$

Therefore, $\bar{\psi}\gamma^5\psi$ is odd under parity.

Part (ii)

Repeating the same calculation,

$$\bar{\psi}\gamma^\mu\gamma^5\psi \rightarrow \psi_p^\dagger \gamma^0 \gamma^\mu \gamma^5 \psi_p \tag{31}$$
$$= |\eta_p|^2 \psi^\dagger \gamma^{0\dagger} \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi \tag{32}$$
$$= \psi^\dagger \gamma^\mu \gamma^5 \gamma^0 \psi \tag{33}$$
$$= -\psi^\dagger \gamma^\mu \gamma^0 \gamma^5 \psi \tag{34}$$

The commutation property of γ^μ and γ^0 can be characterized as

$$\gamma^\mu\gamma^0 = (-1)^\mu\gamma^0\gamma^\mu \tag{35}$$

where $(-1)^\mu = 1$ for $\mu = 0$ and $(-1)^\mu = -1$ for $\mu = 1, 2, 3$.

$$-\psi^\dagger\gamma^\mu\gamma^0\gamma^5\psi = -(-1)^\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^5\psi \tag{36}$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi \to -(-1)^\mu\bar{\psi}\gamma^\mu\gamma^5\psi \tag{37}$$

Therefore, the parity eigenvalue for $\bar{\psi}\gamma^\mu\gamma^5\psi$ is $-(-1)^\mu$.

Part (iii)

The term in the Lagrangian transforms as follows under parity,

$$h\phi\bar{\psi}\left(\frac{1-\gamma^5}{2}\right)\psi + h^*\phi\bar{\psi}\left(\frac{1+\gamma^5}{2}\right)\psi \to h\phi\bar{\psi}\left(\frac{1+\gamma^5}{2}\right)\psi + h^*\phi\bar{\psi}\left(\frac{1-\gamma^5}{2}\right)\psi \tag{38}$$

since the term with γ^5 is odd under parity. Therefore, for parity to be a good symmetry, $h = h^*$, or in other words h should be real.

6.2 - Wick's theorem (Ex 5.3 L & P)

In going from the scalar to fermions, we have to be careful about one fact. In all the definitions of time-ordering and normal-ordering, whenever we have to move a fermionic operator across another one, we introduce a negative sign. This is a matter of definition of the time-ordering and normal-ordering. In this problem we will see that this definition is useful in that it allows us to use Wick's theorem and all of subsequent Feynman diagram technology and makes the definitions consistent with anti-commutations.

Consider the product of two fermion operators. In analogy with the scalar case, we can decompose the fields into the part with creation operator (ψ_-) and the part with annihilation operator (ψ_+).

$$\psi(x)\psi'(x') = \psi_+(x)\psi'_+(x') + \psi_+(x)\psi'_-(x') + \psi_-(x)\psi'_-(x') + \psi_-(x)\psi'_+(x') \quad (39)$$
$$= \psi_+(x)\psi'_+(x') - \psi'_-(x')\psi_+(x) + \{\psi_+(x), \psi'_-(x')\} + \psi_-(x)\psi'_-(x') + \psi_-(x)\psi'_+(x') \quad (40)$$

where we have introduced the anti-commutator to make all the other terms normal-ordered (i.e. to have all the annihilation operators to the right of all the creation operators). Note that the anti-commutator of the two operators is a c-number. Therefore,

$$\{\psi_+(x), \psi'_-(x')\} = \langle 0| \{\psi_+(x), \psi'_-(x')\} |0\rangle \quad (41)$$
$$= \langle 0| \psi_+(x)\psi'_-(x') |0\rangle \quad (42)$$
$$= \langle 0| (\psi_+(x) + \psi_-(x))(\psi'_+(x') + \psi'_-(x')) |0\rangle \quad (43)$$
$$= \langle 0| \psi(x)\psi'(x') |0\rangle \quad (44)$$

where $|0\rangle$ is the vacuum state.

$$\psi(x)\psi'(x') =: \psi(x)\psi'(x'): + \langle 0| \psi(x)\psi'(x') |0\rangle \quad (45)$$

where we have used the following definition of the normal-ordered product.

$$: \psi(x)\psi'(x'): = \psi_+(x)\psi'_+(x') - \psi'_-(x')\psi_+(x) + \psi_-(x)\psi'_-(x') + \psi_-(x)\psi'_+(x') \quad (46)$$

Notice that the normal-ordered product has a negative sign in the second term, when we move a fermion operator around the other.

Now, the time-ordered product for two fermions,

$$\mathcal{T}(\psi(x)\psi'(x')) = \theta(t - t')\psi(x)\psi'(x') - \theta(t' - t)\psi'(x')\psi(x) \quad (47)$$

Therefore,

$$\mathcal{T}(\psi(x)\psi'(x')) = \theta(t - t')\,[: \psi(x)\psi'(x'): + \langle 0| \psi(x)\psi'(x') |0\rangle] - \theta(t' - t)\,[: \psi'(x')\psi(x): + \langle 0| \psi'(x')\psi(x) |0\rangle] \quad (48)$$
$$= \theta(t - t')\,[: \psi(x)\psi'(x'): + \langle 0| \psi(x)\psi'(x') |0\rangle] - \theta(t' - t)\,[-: \psi(x)\psi'(x'): + \langle 0| \psi'(x')\psi(x) |0\rangle] \quad (49)$$
$$=: \psi(x)\psi'(x'): + \langle 0| \theta(t - t')\psi(x)\psi'(x') - \theta(t' - t)\psi'(x')\psi(x) |0\rangle \quad (50)$$
$$=: \psi(x)\psi'(x'): + \langle 0| \mathcal{T}(\psi(x)\psi'(x')) |0\rangle \quad (51)$$

which is the statement of Wick's theorem.