Homework 5 Solutions

5.1 - Anti-commutators of Dirac Fields

We need to prove that $[\psi_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{y},t)]_+ = \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$

We should use the following anticommutation relations.

$$[f_s(\vec{p}), f^\dagger_r(\vec{q})]_+ = [\hat{f}_s(\vec{p}), \hat{f}^\dagger_r(\vec{q})]_+ = \delta_{sr}\,\delta^3(\vec{p}-\vec{q})$$

and $[f_s(\vec{p}), f_r(\vec{q})]_+ = [\hat{f}_s(\vec{p}), \hat{f}_r(\vec{q})]_+ = 0$

$$\psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_s \left[f_s(\vec{p})\, u_{s,\alpha}\, e^{-ip\cdot x} + \hat{f}^\dagger_s(\vec{p})\, v_{s,\alpha}\, e^{ip\cdot x} \right]$$

$$\psi^\dagger_\beta(y) = \int \frac{d^3q}{(2\pi)^{3/2}\sqrt{2E_q}} \sum_r \left[f^\dagger_r(\vec{q})\, u^\dagger_{r,\beta}\, e^{iq\cdot y} + \hat{f}_r(\vec{q})\, v^\dagger_{r,\beta}\, e^{-iq\cdot y} \right]$$

$$\Rightarrow [\psi_\alpha(\vec{x},t), \psi^\dagger_\beta(\vec{y},t)]_+ = \int \frac{d^3p\, d^3q}{(2\pi)^3\sqrt{4E_pE_q}} \sum_{r,s} \Big\{ \delta_{s,r}\,\delta^3(\vec{p}-\vec{q})\, u_{s,\alpha} u^\dagger_{r,\beta}\, e^{iq\cdot y - ip\cdot x}$$

$$+ \delta_{s,r}\,\delta^3(\vec{p}-\vec{q})\, v_{s,\alpha} v^\dagger_{r,\beta}\, e^{ip\cdot x - iq\cdot y} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3\, 2E_p} \sum_r \left\{ u_{r,\alpha} u^\dagger_{r,\beta}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v_{r,\alpha} v^\dagger_{r,\beta}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right\}$$

Recall that $\sum_r u_{r,\alpha} u^\dagger_{r,\beta} = \left[\sum_r u_r u^\dagger_r\right]_{\alpha\beta} = \left[(\not{p}+m)\gamma^0\right]_{\alpha\beta}$

$$\sum_r v_{r,\alpha} v^\dagger_{r,\beta} = \left[\sum_r v_r v^\dagger_r\right]_{\alpha\beta} = \left[(\not{p}-m)\gamma^0\right]_{\alpha\beta}$$

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Therefore:

$$[\psi_\alpha(\vec{x},t),\ \psi^\dagger_\beta(\vec{y},t)]_+ = \int\frac{d^3p}{(2\pi)^3 2E_p}\Big\{[(\not{p}+m)\gamma^0]_{\alpha\beta}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + \cancel{[(\not{p}-m)\gamma^0]_{\alpha\beta}\, e^{-i\vec{p}}}$$

$$+ [(\not{p}-m)\gamma^0]_{\alpha\beta}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big\}$$

Do $\vec{p}\to-\vec{p}$ in the second term

$$\Rightarrow [\psi_\alpha(\vec{x},t),\ \psi^\dagger_\beta(\vec{y},t)]_+ = \int\frac{d^3p}{(2\pi)^3 2E_p}\Big\{\Big[(E_p\gamma^0+\vec{p}\cdot\vec{\gamma}+m)\gamma^0+(E_p\gamma^0-\vec{p}\cdot\vec{\gamma}-m)\gamma^0\Big]_{\alpha\beta} e^{i\vec{p}\cdot(\vec{x}\cdots}$$

$$= \int\frac{d^3p}{(2\pi)^3\cancel{2E_p}}\cdot\cancel{2E_p}(\gamma^0\gamma^0)_{\alpha\beta}\, e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}$$

$$= \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$$

5.2 - Momentum operator

Part (i)

The Dirac Lagrangian,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1}$$

The stress energy tensor for this Lagrangian is,

$$\begin{aligned} T^\mu_\nu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\nu \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}\partial_\nu \bar{\psi} - \mathcal{L}\ \delta^\mu_\nu & (2)\\ &= \bar{\psi} i\gamma^\mu \partial_\nu \psi - \bar{\psi} i \gamma^\mu \partial_\mu \psi\ \delta^\mu_\nu + m\bar{\psi}\psi\ \delta^\mu_\nu & (3)\end{aligned}$$

This does not look symmetric in μ and ν. But just like we showed for the stress-energy tensor for the Lagrangian for electrodynamics in Homework 1, we can add a "total divergence" term to the stress-energy tensor and make it manifestly symmetric. We do not need to do that here.

The linear momentum operator is the conserved charge for spatial translation,

$$\begin{aligned} P^i &= \int d^3x\, T^{0i} & (4)\\ &= -i\int d^3x\ \bar{\psi}\gamma^0 \partial_i \psi & (5)\end{aligned}$$

Remembering that $\bar{\psi} = \psi^\dagger \gamma^0$ and $(\gamma^0)^2 = 1$,

$$P^i = -i\int d^3x\ \psi^\dagger \partial_i \psi \tag{6}$$

Part (ii)

Writing in terms of creation-annihilation operators and expanding,

$$\begin{aligned} P^i = -i\int d^3x\ &\frac{d^3p\ d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}}\sum_{s,s'}(f^\dagger_{s'}(\vec{p'})u^\dagger_{s'}(\vec{p'})e^{ip'\cdot x} + \hat{f}_{s'}(\vec{p'})v^\dagger_{s'}(\vec{p'})e^{-ip'\cdot x})\\ &\qquad \times \partial_i(f_s(\vec{p})u_s(\vec{p})e^{-ip\cdot x} + \hat{f}^\dagger_s(\vec{p})v_s(\vec{p})e^{ip\cdot x}) & (7)\\ = -\int d^3x\ &\frac{d^3p\ d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}}p_i\sum_{s,s'}(f^\dagger_{s'}(\vec{p'})u^\dagger_{s'}(\vec{p'})e^{ip'\cdot x} + \hat{f}_{s'}(\vec{p'})v^\dagger_{s'}(\vec{p'})e^{-ip'\cdot x})\\ &\qquad \times (-f_s(\vec{p})u_s(\vec{p})e^{-ip\cdot x} + \hat{f}^\dagger_s(\vec{p})v_s(\vec{p})e^{ip\cdot x}) & (8)\\ = -\int &\frac{d^3p\ d^3p'}{(2\pi)^3\sqrt{2E_p 2E_{p'}}}p_i\sum_{s,s'}\int d^3x\ (-f^\dagger_{s'}(\vec{p'})u^\dagger_{s'}(\vec{p'})f_s(\vec{p})u_s(\vec{p})e^{-i(p-p')\cdot x}\\ &\qquad + f^\dagger_{s'}(\vec{p'})u^\dagger_{s'}(\vec{p'})\hat{f}^\dagger_s(\vec{p})v_s(\vec{p})e^{i(p+p')\cdot x}\\ &\qquad - \hat{f}_{s'}(\vec{p'})v^\dagger_{s'}(\vec{p'})f_s(\vec{p})u_s(\vec{p})e^{-i(p+p')\cdot x}\\ &\qquad + \hat{f}_{s'}(\vec{p'})v^\dagger_{s'}(\vec{p'})\hat{f}^\dagger_s(\vec{p})v_s(\vec{p})e^{i(p-p')\cdot x}) & (9)\end{aligned}$$

The x-integration over the plane-wave factors yields δ-functions for the momenta, which helps get rid of one momentum integration,

$$P^i = -\int \frac{d^3p}{2E_p} p_i \sum_{s,s'} (-f^\dagger_{s'}(\vec{p}) u^\dagger_{s'}(\vec{p}) f_s(\vec{p}) u_s(\vec{p}) + f^\dagger_{s'}(-\vec{p}) u^\dagger_{s'}(-\vec{p}) \hat{f}^\dagger_s(\vec{p}) v_s(\vec{p}) \\ - \hat{f}_{s'}(-\vec{p}) v^\dagger_{s'}(-\vec{p}) f_s(\vec{p}) u_s(\vec{p}) + \hat{f}_{s'}(\vec{p}) v^\dagger_{s'}(\vec{p}) \hat{f}^\dagger_s(\vec{p}) v_s(\vec{p})) \tag{10}$$

We can now use the normalization conditions for spinor coefficients,

$$u^\dagger_r(\vec{p}) u_s(\vec{p}) = v^\dagger_r(\vec{p}) v_s(\vec{p}) = 2E_p \delta_{rs} \tag{11}$$
$$v^\dagger_r(\vec{p}) u_s(-\vec{p}) = u^\dagger_r(\vec{p}) v_s(-\vec{p}) = 0 \tag{12}$$

This kills off the second and third term in the expression above. Thus,

$$P_i = \int d^3p\ p_i \sum_s (f^\dagger_s(\vec{p}) f_s(\vec{p}) - \hat{f}_s(\vec{p}) \hat{f}^\dagger_s(\vec{p})) \tag{13}$$
$$= \int d^3p\ p_i \sum_s (f^\dagger_s(\vec{p}) f_s(\vec{p}) + \hat{f}^\dagger_s(\vec{p}) \hat{f}_s(\vec{p}) - \{\hat{f}^\dagger_s(\vec{p}), \hat{f}_s(\vec{p})\}) \tag{14}$$

The anti-commutator of the creation-annihilation operators is symmetric in $\vec{p}$, so that term multiplied with p_i drops out on integration. This appeared earlier on the homework, when we concluded that we do not need to normal order the momentum operator (unlike the Hamiltonian, where the vacuum has an infinite energy, and hence has to be normal ordered away).

$$P_i = \int d^3p\ p_i \sum_s (f^\dagger_s(\vec{p}) f_s(\vec{p}) + \hat{f}^\dagger_s(\vec{p}) \hat{f}_s(\vec{p})) \tag{15}$$

Part (iii)

To calculate the momentum of a given state, we act on it with the momentum operator,

$$P_i\ f^\dagger_s(\vec{p}) |0\rangle = \int d^3p'\ p'_i \sum_{s'} (f^\dagger_{s'}(\vec{p'}) f_{s'}(\vec{p'}) + \hat{f}^\dagger_{s'}(\vec{p'}) \hat{f}_{s'}(\vec{p'}))\ f^\dagger_s(\vec{p}) |0\rangle \tag{16}$$

Remember, f and $\hat{f}$ anti-commute, so we can pay a negative sign and flip the order of f and $\hat{f}$ whenever we want.

$$P_i\ f^\dagger_s(\vec{p}) |0\rangle = \int d^3p'\ p'_i \sum_{s'} f^\dagger_{s'}(\vec{p'}) f_{s'}(\vec{p'})\ f^\dagger_s(\vec{p}) |0\rangle + \hat{f}^\dagger_{s'}(\vec{p'}) \hat{f}_{s'}(\vec{p'})\ f^\dagger_s(\vec{p}) |0\rangle \tag{17}$$
$$= \int d^3p'\ p'_i \sum_{s'} f^\dagger_{s'}(\vec{p'}) f_{s'}(\vec{p'})\ f^\dagger_s(\vec{p}) |0\rangle - \hat{f}^\dagger_{s'}(\vec{p'})\ f^\dagger_s(\vec{p}) \hat{f}_{s'}(\vec{p'}) |0\rangle \tag{18}$$
$$= \int d^3p'\ p'_i \sum_{s'} f^\dagger_{s'}(\vec{p'}) f_{s'}(\vec{p'})\ f^\dagger_s(\vec{p}) |0\rangle \tag{19}$$

Now, using the anti-commutator for f with f,

$$P_i\ f_s^\dagger(\vec{p})\,|0\rangle = \int d^3p'\ p'_i \sum_{s'} f_{s'}^\dagger(\vec{p'})(\{f_{s'}(\vec{p'}),\ f_s^\dagger(\vec{p})\} -\ f_s^\dagger(\vec{p})\ f_{s'}(\vec{p'}))\,|0\rangle \tag{20}$$

$$= \int d^3p'\ p'_i \sum_{s'} \delta_{ss'}\delta^3(\vec{p}-\vec{p'})f_{s'}^\dagger(\vec{p'})\,|0\rangle \tag{21}$$

$$= p_i\ f_s^\dagger(\vec{p})\,|0\rangle \tag{22}$$

Therefore, the given state is an eigenstate of the momentum operator with the eigenvalue p_i.

Part (iv)

We can repeat the same steps for the anti-particle state. It is clear from the form of the momentum operator that $f \to \hat{f}$ leaves the momentum operator invariant. Thus, all the previous steps are identical for this case.

$$P_i\ \hat{f}_s^\dagger(\vec{p})\,|0\rangle = p_i\ \hat{f}_s^\dagger(\vec{p})\,|0\rangle \tag{23}$$

This sign is opposite to the "sign" of the v_s spinor. The plane wave $e^{+ip\cdot x}$ describes a propagation with momentum $-\vec{p}$.

5.3 - Dirac propagator

5.3.1 - Formula for Green's function

Problem 5.2.1 / Ex 4.29 of L&P

$$iS_F(x-x') = i(i\not\partial + m)\Delta_F(x-x')$$
$$= (i\not\partial + m)\int \frac{d^3p}{(2\pi)^3 2E_p}\left[\theta(t-t')e^{-ip\cdot(x-x')} + \theta(t'-t)e^{ip\cdot(x-x')}\right]$$

Note that $\vec{\gamma}\cdot\vec{\nabla}\left[\theta(t-t')e^{-ip\cdot(x-x')} + \theta(t'-t)e^{ip\cdot(x-x')}\right]$

$$= \left[\theta(t-t')(+i\vec{\gamma}\cdot\vec{p})e^{-ip\cdot(x-x')} + \theta(t'-t)(-i\vec{\gamma}\cdot\vec{p})e^{ip\cdot(x-x')}\right]$$

and
$$\gamma^0\partial_t\left[\theta(t-t')e^{-ip\cdot(x-x')} + \theta(t'-t)e^{ip\cdot(x-x')}\right]$$
$$= \gamma^0\delta(t-t')e^{-ip\cdot(x-x')} - \gamma^0\delta(t-t')e^{ip\cdot(x-x')} + \gamma^0\left[\theta(t-t')(-iE_p)e^{-ip\cdot(x-x')} + \theta(t'-t)(iE_p) \times e^{ip\cdot(x-x')}\right]$$

where we used $\partial_t\theta(t-t') = \delta(t-t')$

③

$$\Rightarrow iS_F(x-x') = \int\frac{d^3p}{(2\pi)^3 2E_p}\Big[\theta(t-t')e^{-ip\cdot(x-x')}\left((E_p\gamma^0 - \vec{\gamma}\cdot\vec{p}) + m\right)$$
$$+ \theta(t'-t)e^{ip\cdot(x-x')}(-E_p\gamma^0 + \vec{\gamma}\cdot\vec{p} + m)$$
$$+ \gamma^0\delta(t-t')\left(e^{-ip\cdot(x-x')} - e^{ip\cdot(x-x')}\right)\Big]$$

It is easy to see that the last term vanishes when we do $\vec{p}\to-\vec{p}$.

Therefore: $$iS_F(x-x') = \int\frac{d^3p}{(2\pi)^3 2E_p}\left[\theta(t-t')e^{-ip\cdot(x-x')}(\not p + m) - \theta(t'-t)e^{ip\cdot(x-x')}(\not p - m)\right]$$

□

5.3.2 - Relating Green's function to fields

problem 5.2.2/Ex 4.30 of L&P

We have $\psi_\alpha(x) = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_s \left(f_s(\vec{p})\, u_{s,\alpha}\, e^{-ip\cdot x} + \hat{f}_s^{\dagger}(\vec{p})\, v_{s,\alpha}\, e^{ip\cdot x} \right)$

$\bar{\psi}_\beta(x') = \int \frac{d^3q}{(2\pi)^{3/2}\sqrt{2E_q}} \sum_r \left(f_r^{\dagger}(\vec{q})\, \bar{u}_{r,\beta}\, e^{iq\cdot x'} + \hat{f}_r(\vec{q})\, \bar{v}_{r,\beta}\, e^{-iq\cdot x'} \right)$

$\Rightarrow \langle 0| \psi_\alpha(x)\, \bar{\psi}_\beta(x') |0\rangle = \int \frac{d^3q\, d^3p}{(2\pi)^3\sqrt{4E_pE_q}} \sum_{s,r} \langle 0| f_s(\vec{p}) f_r^{\dagger}(\vec{q}) |0\rangle\, u_{s,\alpha} \bar{u}_{r,\beta}\, e^{iq\cdot x' - ip\cdot x}$

Recall $[f_s(\vec{p}), f_r^{\dagger}(\vec{q})]_+ = \delta_{r,s}\, \delta^3(\vec{p}-\vec{q})$

$\Rightarrow \langle 0| \psi_\alpha(x)\, \bar{\psi}_\beta(x') |0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_r u_{r,\alpha} \bar{u}_{r,\beta}\, e^{ip\cdot(x'-x)}$

$= \int \frac{d^3p}{(2\pi)^3 2E_p} \cdot (\not{p}+m)_{\alpha\beta}\, e^{ip\cdot(x'-x)}$

On the other hand

$\langle 0| \bar{\psi}_\beta(x')\, \psi_\alpha(x) |0\rangle = \int \frac{d^3p}{(2\pi)^3 2E_p} \sum_s v_{s,\alpha} \bar{v}_{s,\beta}\, e^{-iq\cdot x' + ip\cdot x}$

$= \int \frac{d^3p}{(2\pi)^3 2E_p} (\not{p}-m)_{\alpha\beta}\, e^{ip\cdot(x-x')}$

Therefore $\langle 0| \mathcal{T}[\psi_\alpha(x)\, \bar{\psi}_\beta(x')] |0\rangle$

$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[\theta(t-t')(\not{p}+m)_{\alpha\beta}\, e^{-ip\cdot(x-x')} - \theta(t'-t)(\not{p}-m)_{\alpha\beta}\, e^{ip\cdot(x-x')} \right]$

$= i\, S_{F\alpha\beta}(x-x')$ $\square$

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5.4 - Parity transformation of Dirac and scalar fields

5.4.1 - Fermion and anti-fermion states have opposite (intrinsic) parity

In Dirac-Pauli representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$u_\pm(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\pm \end{pmatrix} \quad ; \quad v_\pm(\vec{p}) = \pm\sqrt{E_p+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\mp \\ \chi_\mp \end{pmatrix}$$

$$\Rightarrow \gamma^0 u_\pm(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\pm \end{pmatrix} = u_\pm(-\vec{p})$$

$$\gamma^0 v_\pm(\vec{p}) = \pm\sqrt{E_p+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\mp \\ -\chi_\mp \end{pmatrix} = \mp\sqrt{E_p+m}\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\mp \\ \chi_\mp \end{pmatrix} = -v_\pm(-\vec{p})$$

(5)

under parity transformation, we have

$$\psi(x) \xrightarrow{P} \psi_p(x) = \eta_p \gamma_0 \psi(\tilde{x})$$

$$= \eta_p \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_s \left(f_s(\vec{p})\, u_s(-\vec{p})\, e^{-ip\cdot\tilde{x}} - \hat{f}_s^{\dagger}(\vec{p})\, v_s(-\vec{p})\, e^{ip\cdot\tilde{x}} \right)$$

$$\overset{\vec{p}\to-\vec{p}}{=} \eta_p \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_s \left(f_s(-\vec{p})\, u_s(\vec{p})\, e^{-ip\cdot x} - \hat{f}_s^{\dagger}(-\vec{p})\, v_s(\vec{p})\, e^{ip\cdot x} \right)$$

Therefore under parity transformation

$$f_s(\vec{p}) \xrightarrow{P} \eta_p f_s(-\vec{p})$$

$$\hat{f}_s^{\dagger}(\vec{p}) \xrightarrow{P} -\eta_p \hat{f}_s^{\dagger}(-\vec{p})$$

Therefore, fermion and anti-fermion states have opposite intrinsic parity. □

5.4.2 - "Selection rules"

(i) parity of initial state $P_\phi = +1$

parity of final state $P_{f\bar{f}} = (-1)\cdot(-1)^L = (-1)^{L+1}$

$\Rightarrow$ L has to be odd.

(ii) Initial angular momentum $J_\phi = 0$

Final angular momentum $J_{f\bar{f}} = L_{f\bar{f}} + S_{f\bar{f}}$.

Since fermions have spin $= \frac{1}{2}$ $\Rightarrow$ $S_{f\bar{f}} = 0, 1$

Based on angular momentum conservation, $L_{f\bar{f}} = 0$ or 1, but $L_{f\bar{f}} = 0$ is forbidden by parity. Therefore $L_{f\bar{f}} = 1$.

(iii) From Bose statistics, interchange of $\phi\phi$ should be even

$\Rightarrow (-1)^{L_{\phi\phi}} = 1 \Rightarrow L_{\phi\phi}$ is even

(iv) Initial state parity: $(-1)\cdot(-1)^{L_{f\bar{f}}} = (-1)^{1+L_{f\bar{f}}} = 1$

$\Rightarrow$ $L_{f\bar{f}}$ for $f\bar{f} \to \phi\phi$ should be odd.

(v) Initial angular momentum: $J_{f\bar{f}} = L_{f\bar{f}} + S_{f\bar{f}}$

Final angular momentum: $J_{\phi\phi} = L_{\phi\phi}$

since $L_{f\bar{f}}$ is odd while $L_{\phi\phi}$ is even

$\Rightarrow$ $S_{f\bar{f}}$ is odd.

However $S_{f\bar{f}} = 0, 1$. Therefore $S_{f\bar{f}}$ can only be 1.