Homework 4 Solutions

4.1 - Weyl or Chiral representation for γ-matrices

4.1.1: Anti-commutation relations

We can write out the γ^μ matrices as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

where

$$\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma}), \qquad \bar{\sigma}^\mu = (\mathbb{1}_2, -\boldsymbol{\sigma})$$

The anticommutator is

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu \end{pmatrix} + \begin{pmatrix} \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \\
&= \begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}
\end{aligned}$$

Consider the upper-left component, $\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu$. For $\mu = \nu = 0$,

$$\sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0 = 2 \times \mathbb{1}_2$$

For $\mu = 0$ and $\nu \neq 0$,

$$\sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0 = 0$$

For $\mu \neq 0$ and $\nu \neq 0$, we get

$$\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i = -\sigma^i\sigma^j - \sigma^j\sigma^i = -\left\{\sigma^i, \sigma^j\right\} = -2\delta^{ij}$$

Putting all of these together, we get

$$\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu} \times \mathbb{1}_2$$

In exactly the same way,

$$\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu = 2g^{\mu\nu} \times \mathbb{1}_2$$

so

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \times \mathbb{1}_4$$

4.1.2: Boost and rotation generators

We can write

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4}(\{\gamma^\mu, \gamma^\nu\} - 2\gamma^\nu\gamma^\mu) = \frac{i}{2}(g^{\mu\nu} - \gamma^\nu\gamma^\mu) \tag{1}$$

And,

$$\begin{aligned}\gamma^\nu\gamma^\mu &= \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}\end{aligned}$$

$$S^{\mu\nu} = \frac{i}{2}\begin{pmatrix} g^{\mu\nu} - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & g^{\mu\nu} - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \tag{2}$$

Then,

$$K^i = S^{0i} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{3}$$

$$\begin{aligned}
J^k &= \frac{1}{2}\epsilon^{ijk}S^{ij} \qquad (4)\\
&= \frac{1}{2}\epsilon^{ijk}\frac{i}{2}\begin{pmatrix} g^{ij} - \sigma^j\bar{\sigma}^i & 0 \\ 0 & g^{ij} - \bar{\sigma}^j\sigma^i \end{pmatrix} \qquad (5)\\
&= \frac{i}{4}\epsilon^{ijk}\begin{pmatrix} \sigma^j\sigma^i & 0 \\ 0 & \sigma^j\sigma^i \end{pmatrix} \qquad (6)\\
&= \frac{i}{8}\epsilon^{ijk}\begin{pmatrix} [\sigma^j, \sigma^i] & 0 \\ 0 & [\sigma^j, \sigma^i] \end{pmatrix} \qquad (7)\\
&= \frac{i}{8}\epsilon^{ijk}\begin{pmatrix} 2i\epsilon^{jim\sigma^m} & 0 \\ 0 & 2i\epsilon^{jim}\sigma^m \end{pmatrix} \qquad (8)\\
&= \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \qquad (9)
\end{aligned}$$

4.2 - General representation of γ-matrices

4.2.1: Lorentz group algebra

In order to have a compact notation, let us evaluate the following,

$$[S^{\mu\nu}, S^{\rho\sigma}] \tag{10}$$

We look at this because the Lorentz generators are made out of $S^{\mu\nu}$,and their commutation will follow from the quantity above. We can write

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4}(\{\gamma^\mu, \gamma^\nu\} - 2\gamma^\nu\gamma^\mu) = \frac{i}{2}(g^{\mu\nu} - \gamma^\nu\gamma^\mu) \tag{11}$$

Now, since $g^{\mu\nu}$ is implicitly multiplied with the identity spinor space, the commutator we are after is

$$[S^{\mu\nu}, S^{\rho\sigma}] = -\tfrac{1}{4}[g^{\mu\nu} - \gamma^\nu\gamma^\mu, g^{\rho\sigma} - \gamma^\sigma\gamma^\rho] = \tfrac{1}{4}[\gamma^\sigma\gamma^\rho, \gamma^\nu\gamma^\mu]$$

The strategy to evaluate this commutator is roughly as follows. We keep anti-commuting the γ-matrices in the first term, till we get the second term. Each anti-commutation gives us something in the form $g^{\mu\nu}\gamma^\rho\gamma^\sigma$. We collect all these terms in the end, and rewrite them in terms of $S^{\mu\nu}$. Carrying out the calculation,

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= \tfrac{1}{4}(\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}((2g^{\sigma\rho} - \gamma^\rho\gamma^\sigma)\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - \gamma^\rho\gamma^\sigma\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - \gamma^\rho(2g^{\sigma\nu} - \gamma^\nu\gamma^\sigma)\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + \gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + \gamma^\rho\gamma^\nu(2g^{\sigma\mu} - \gamma^\mu\gamma^\sigma) - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - (2g^{\rho\nu} - \gamma^\nu\gamma^\rho)\gamma^\mu\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\nu}\gamma^\mu\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\nu}\gamma^\mu\gamma^\sigma \\
&\qquad + \gamma^\nu(2g^{\rho\mu} - \gamma^\mu\gamma^\rho)\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\nu}\gamma^\mu\gamma^\sigma + 2g^{\rho\mu}\gamma^\nu\gamma^\sigma \\
&\qquad - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\nu}\gamma^\mu\gamma^\sigma + 2g^{\rho\mu}\gamma^\nu\gamma^\sigma \\
&\qquad - \gamma^\nu\gamma^\mu(2g^{\rho\sigma} - \gamma^\sigma\gamma^\rho) - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= \tfrac{1}{4}(2g^{\sigma\rho}\gamma^\nu\gamma^\mu - 2g^{\sigma\nu}\gamma^\rho\gamma^\mu + 2g^{\sigma\mu}\gamma^\rho\gamma^\nu - 2g^{\rho\nu}\gamma^\mu\gamma^\sigma + 2g^{\rho\mu}\gamma^\nu\gamma^\sigma - 2g^{\rho\sigma}\gamma^\nu\gamma^\mu \\
&\qquad + \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\sigma\gamma^\rho) \\
&= -\tfrac{1}{2}(g^{\nu\sigma}\gamma^\rho\gamma^\mu - g^{\mu\sigma}\gamma^\rho\gamma^\nu + g^{\nu\rho}\gamma^\mu\gamma^\sigma - g^{\mu\rho}\gamma^\nu\gamma^\sigma)
\end{aligned}$$

Now we can add $g^{\nu\sigma}g^{\rho\mu} - g^{\mu\rho}g^{\sigma\nu}$ and $g^{\nu\rho}g^{\sigma\mu} - g^{\mu\sigma}g^{\nu\rho}$:

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] = i[&-\tfrac{i}{2}(g^{\mu\rho} - \gamma^\rho\gamma^\mu)g^{\nu\sigma} + \tfrac{i}{2}(g^{\nu\rho} - \gamma^\rho\gamma^\nu)g^{\mu\sigma} \\
&- \tfrac{i}{2}(g^{\sigma\mu} - \gamma^\mu\gamma^\sigma)g^{\nu\rho} + \tfrac{i}{2}(g^{\sigma\nu} - \gamma^\nu\gamma^\sigma)g^{\mu\rho}]
\end{aligned}$$

Using the above and the fact that $S^{\mu\nu}$ is antisymmetric, we get

$$\boxed{[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})}$$

In principle, we are done already, because one can show that this is the same commutation relation that the $\mathcal{J}^{\mu\nu}$ matrices (defined in Problem 4.2.2) satisfy, and hence $S^{\mu\nu}$ satisfies the same commutation relation as Lorentz transformation generator.

However, let us calculate the commutators explicitly in terms of J^i, K^i etc.

$$\begin{aligned}
[J^i, J^j] &= \frac{1}{4}\epsilon^{mni}\epsilon^{pqj}[S^{mn}, S^{pq}] && (12)\\
&= \frac{i}{4}\epsilon^{mni}\epsilon^{pqj}(g^{np}S^{mq} - g^{mp}S^{nq} - g^{nq}S^{mp} + g^{mq}S^{np}) && (13)\\
&= -i\epsilon^{mni}\epsilon^{pqj}g^{mp}S^{nq} && (14)\\
&= i\epsilon^{mni}\epsilon^{mqj}S^{nq} && (15)\\
&= i\left(\delta_{nq}\delta_{ij} - \delta_{jn}\delta_{iq}\right)S^{nq} && (16)\\
&= -i\,\delta_{jn}\delta_{iq}\,S^{nq} && (17)\\
&= -\frac{i}{2}\,(\delta_{jn}\delta_{iq} - \delta_{jq}\delta_{in})S^{nq} && (18)\\
&= \frac{i}{2}\,\epsilon^{ijk}\epsilon^{nqk}S^{nq} && (19)\\
&= i\epsilon^{ijk}J^k && (20)
\end{aligned}$$

where we have used the ϵ-tensor contraction identity

$$\epsilon^{ijk}\epsilon^{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \tag{21}$$

and the anti-symmetry of $S^{\mu\nu}$ in the above derivation.

The other two commutation relations follow from similar manipulations.

$$\begin{aligned}
[K^i, K^j] &= [S^{0i}, S^{0j}] = -iS^{ij} = -i\epsilon^{ijk}J^k && (22)\\
[K^i, J^j] &= \frac{1}{2}\epsilon^{mnj}[S^{0i}, S^{mn}] && (23)\\
&= \frac{i}{2}\epsilon^{mnj}(g^{im}S^{0n} - g^{in}S^{0m}) && (24)\\
&= -i\epsilon^{inj}S^{0n} && (25)\\
&= i\epsilon^{ijk}J^k && (26)
\end{aligned}$$

4.2.2: Meaning of μ index on γ^μ

We have the commutator

$$\begin{aligned}
[\gamma^\mu, S^{\rho\sigma}] &= \frac{i}{2}[\gamma^\mu, g^{\rho\sigma} - \gamma^\sigma\gamma^\rho] \\
&= -\frac{i}{2}[\gamma^\mu, \gamma^\sigma\gamma^\rho] \\
&= -\frac{i}{2}\gamma^\sigma[\gamma^\mu, \gamma^\rho] + [\gamma^\mu, \gamma^\sigma]\gamma^\rho \\
&= -i\{\gamma^\sigma(g^{\mu\rho} - \gamma^\rho\gamma^\mu) + (g^{\mu\sigma} - \gamma^\sigma\gamma^\mu)\gamma^\rho\} \\
&= -ig^{\mu\rho}\gamma^\sigma - ig^{\mu\sigma}\gamma^\rho + i\gamma^\sigma\gamma^\rho\gamma^\mu + i\gamma^\sigma\gamma^\mu\gamma^\rho \\
&= -ig^{\mu\rho}\gamma^\sigma - ig^{\mu\sigma}\gamma^\rho + 2ig^{\mu\rho}\gamma^\sigma - i\gamma^\sigma\gamma^\mu\gamma^\rho + i\gamma^\sigma\gamma^\mu\gamma^\rho \\
&= ig^{\mu\rho}\gamma^\sigma - ig^{\mu\sigma}\gamma^\rho
\end{aligned}$$

We can write the right-hand side down in the same form by substituting the explicit representation of $(\mathcal{J}^{\rho\sigma})^\mu_\nu$:

$$\begin{aligned}
(\mathcal{J}^{\rho\sigma})^\mu_\nu\gamma^\nu &= ig^{\mu\alpha}(\delta^\rho_\alpha\delta^\sigma_\nu - \delta^\sigma_\alpha\delta^\rho_\nu)\gamma^\nu \\
&= ig^{\mu\alpha}\delta^\rho_\alpha\delta^\sigma_\nu\gamma^\nu - ig^{\mu\alpha}\delta^\sigma_\alpha\delta^\rho_\nu\gamma^\nu \\
&= ig^{\mu\rho}\gamma^\sigma - ig^{\mu\sigma}\gamma^\rho
\end{aligned}$$

Thus,

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu_\nu\gamma^\nu$$

4.2.3: Chirality projection operator

Following the steps in Problem 4.2.1, we can write,

$$S^{\mu\nu} = \frac{i}{2}(g^{\mu\nu} - \gamma^\mu\gamma^\nu) \tag{27}$$

Note that γ^5 anti-commutes with all the γ^μ.

$$\{\gamma^5, \gamma^\mu\} = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3) \tag{28}$$

For a given value of μ, we can anti-commute the γ^μ in each term all the way to the corresponding γ in γ^5. In each anti-commutation, we pick up a negative sign. There are even anti-commutations in one term, and odd in the other, and thus they always cancel. Let us do the steps for $\mu = 1$.

$$\begin{aligned}
\{\gamma^5, \gamma^1\} &= i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1 + \gamma^1\gamma^0\gamma^1\gamma^2\gamma^3) && (29) \\
&= i((-1)^2\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3 + (-1)\gamma^0\gamma^1\gamma^1\gamma^2\gamma^3) && (30) \\
&= 0 && (31)
\end{aligned}$$

Therefore,

$$[\gamma^5, S^{\mu\nu}] = \frac{i}{2}[\gamma^5, (g^{\mu\nu} - \gamma^\mu\gamma^\nu)] \tag{32}$$
$$= -\frac{i}{2}[\gamma^5, \gamma^\mu\gamma^\nu] \tag{33}$$
$$= -\frac{i}{2}[\gamma^5, \gamma^\mu]\gamma^\nu + \gamma^\mu[\gamma^5, \gamma^\nu] \tag{34}$$
$$= -\frac{i}{2}[\gamma^5, \gamma^\mu]\gamma^\nu + \gamma^\mu[\gamma^5, \gamma^\nu] \tag{35}$$
$$= -i(\gamma^5\gamma^\mu\gamma^\nu + \gamma^\mu\gamma^5\gamma^\nu) \tag{36}$$
$$= -i(\gamma^5\gamma^\mu\gamma^\nu - \gamma^5\gamma^\mu\gamma^\nu) = 0 \tag{37}$$

4.3 - Lorentz Transformations

4.3.1: Modified spinor (Ex 4.6 Lahiri and Pal)

Please ignore the text above the line in the scan.

(4)

Recall that $\{\sigma^i, \sigma^j\}_+ = 2\delta^{ij}$

$\Rightarrow \quad \{\gamma^i, \gamma^j\}_+ = -2\,\mathbb{1}_4$.

Therefore. ~~$\{\gamma^\mu, \gamma^j\}_+ = 2g$~~

$$\{\gamma^\mu, \gamma^\nu\}_+ = 2g^{\mu\nu}\,\mathbb{1}_4$$

$$(\gamma^0)^\dagger = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \gamma^0 = \gamma^0\gamma^0\gamma^0$$

$$(\gamma^i)^\dagger = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\gamma^0\gamma^i\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\Rightarrow \quad (\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

□.

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problem 4.2.1 / L&P Ex 4.6

Under Lorentz transformation

$$\psi \to \psi' = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}}\psi$$

$$\Rightarrow \quad \psi_c \to \psi_c' = C\gamma_0^T e^{\frac{i}{4}\sigma^*_{\mu\nu}\omega^{\mu\nu}}\psi^*$$

$$= C\gamma_0^T e^{\frac{i}{4}\sigma^*_{\mu\nu}\omega^{\mu\nu}}(C\gamma_0^T)^{-1}(C\gamma_0^T)\psi^*$$

$$= \exp\left[\frac{i}{4}(C\gamma_0^T)(\sigma^*_{\mu\nu}\omega^{\mu\nu})(C\gamma_0^T)^{-1}\right]\cdot(C\gamma_0^T)\psi^*$$

(5)

recall that

$$(\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0$$

$$\Rightarrow \quad \gamma_\mu^* = \gamma^{0T}\gamma^{\mu T}\gamma^{0T}$$

$$\Rightarrow \quad \sigma_{\mu\nu}^* = -\frac{i}{2}[\gamma_\mu^*, \gamma_\nu^*]$$

$$= -\frac{i}{2}\gamma^{0T}[\gamma_\mu^T, \gamma_\nu^T]\gamma^{0T}$$

$$= \gamma^{0T}\sigma_{\mu\nu}^T\gamma^{0T}$$

Also note $\gamma_0^T = \gamma_0 = \gamma_0^{-1}$ and $\gamma_0^2 = \mathbb{1}$

$$\Rightarrow \quad (C\gamma_0^T)(\sigma_{\mu\nu}^* \omega^{\mu\nu})(C\gamma_0^T)^{-1}$$

$$= C\sigma_{\mu\nu}^T C^{-1}\omega^{\mu\nu}$$

(use $C\gamma_\mu^T C^{-1} = -\gamma_\mu$)

$$= -\frac{i}{2}\omega^{\mu\nu} C[\gamma_\mu^T, \gamma_\nu^T]C^{-1}$$

$$= -\frac{i}{2}\omega^{\mu\nu}[\gamma_\mu, \gamma_\nu]$$

$$= -\omega^{\mu\nu}\sigma_{\mu\nu}$$

Therefore $\psi_c' = \exp[-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}]\psi_c$, hence have the same transformation properties as ψ.

4.3.2: Bilinears

Part (i) Ex 4.5 Lahiri and Pal

(6)

• problem 4.2.2 / L&P Ex. 4.5

ψ transform under Lorentz transformation as

$$\psi \to \psi' = S(\Lambda)\psi \quad , \text{where } S(\Lambda) = e^{-\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}}$$

$\bar{\psi}$ transform as :

$$\bar{\psi} \to \bar{\psi}' = \psi^{+} e^{\frac{i}{4}\sigma_{\mu\nu}^{+}\omega^{\mu\nu}} \gamma^{0}$$

recall that $\quad \gamma^0 (\gamma^\mu)^{+} \gamma^0 = \gamma^\mu \Rightarrow \quad \gamma^0 \sigma_{\mu\nu}^{+} \gamma^0 = \sigma_{\mu\nu}$

$$\Rightarrow \bar{\psi}' = \psi^{+} \gamma^0 \gamma^0 e^{\frac{i}{4}\sigma_{\mu\nu}^{+}\omega^{\mu\nu}} \gamma^0$$

$$= \psi^{+}\gamma^0 \; e^{\frac{i}{4}\gamma^0 \sigma_{\mu\nu}^{+}\omega^{\mu\nu}\gamma^0}$$

$$= \bar{\psi}\, e^{\frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}}$$

$$= \bar{\psi}\; S(\Lambda)^{-1}$$

Therefore $\bar{\psi}\psi \to \bar{\psi}'\psi' = \bar{\psi}\psi$, i.e., $\bar{\psi}\psi$ transform as a scalar.

To get transformation properties of $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\sigma^{\mu\nu}\psi$, we recall

$$S^{-1}(\Lambda)\gamma^\mu \Lambda^{\nu}_{\mu} S(\Lambda) = \gamma^\nu \qquad (\text{Eq } 4.30 \text{ in L\&P})$$

since $\Lambda^{\alpha}{}_{\beta}\Lambda^{\beta}{}_{\gamma} = \delta^{\alpha}{}_{\gamma}$, we can rewrite the above equation as.

$$S^{-1}(\Lambda)\, \gamma^\mu \, S(\Lambda) = \gamma^\nu \Lambda^{\mu}_{\nu}$$

Then $\bar{\psi}\gamma^\mu\psi \to \bar{\psi}S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi = \bar{\psi}\gamma^\nu\psi\,\Lambda^{\mu}_{\nu}$

and $\bar{\psi}\sigma^{\mu\nu}\psi \to \frac{i}{2}\bar{\psi}S^{-1}(\Lambda)[\gamma^\mu,\gamma^\nu]S(\Lambda)\psi = \bar{\psi}\sigma^{\rho\sigma}\psi\,\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}$

Therefore $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\sigma^{\mu\nu}\psi$ transform as vector and tensor respectively. 17

Part (ii)

We have that

$$\psi^\dagger \to \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger$$

In infinitesimal form, this reads

$$\psi^\dagger \to \psi^\dagger \left[1 + \tfrac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]$$

If we define

$$\bar{\psi} := \psi^\dagger \gamma^0$$

we have the transformation

$$\bar{\psi} \to \psi^\dagger \left[1 + \tfrac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right]\gamma^0$$

Since S^{ij} is given by

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$$

we have $(S^{ij})^\dagger = (S^{ij})$. Also,

$$\begin{aligned}
[\gamma^0, S^{ij}] &= \frac{1}{2}\epsilon^{ijk}\left[\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix} - \begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\right] \\
&= \frac{1}{2}\epsilon^{ijk}\left[\begin{pmatrix}0 & \sigma^k \\ \sigma^k & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma^k \\ \sigma^k & 0\end{pmatrix}\right] \\
&= 0
\end{aligned}$$

In terms with μ or ν zero, we have

$$\begin{aligned}
\left(S^{0i}\right) &= -\frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}, \\
\left(S^{0i}\right)^\dagger &= \frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix} = -\left(S^{0i}\right)
\end{aligned}$$

Now consider the anti-commutator,

$$\begin{aligned}
\left\{\gamma^0, S^{0i}\right\} &= -\frac{i}{2}\left[\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix} + \begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\right] \\
&= \frac{i}{2}\left[\begin{pmatrix}0 & -\sigma^i \\ \sigma^i & 0\end{pmatrix} + \begin{pmatrix}0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}\right] \\
&= 0
\end{aligned}$$

Therefore,

$$(S^{\mu\nu})^\dagger \gamma^0 = \gamma^0 (S^{\mu\nu})$$

and the transformation for $\bar{\psi}$ becomes

$$\psi^\dagger\gamma^0 \to \psi^\dagger\gamma^0\left[1 + \tfrac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})\right]$$

For finite rotations, this is just

$$\bar{\psi} \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}$$

From this relationship, we immediately see that

$$\bar{\psi}\psi \to \bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\psi$$

so

$$\boxed{\bar{\psi}\psi \to \bar{\psi}\psi}$$

that is, $\bar{\psi}\psi$ transforms like a Lorentz scalar.

4.4 - Gordon identity

We begin with the following two equations,

$$(\not{p} - m)u(p) = 0 \tag{38}$$
$$\bar{u}(p')(\not{p}' - m) = 0 \tag{39}$$

Multiplying with appropriate spinors and adding these two together,

$$\bar{u}(p')\gamma^\mu(\not{p} - m)u(p) + \bar{u}(p')(\not{p}' - m)\gamma^\mu u(p) = 0 \tag{40}$$
$$\Rightarrow \quad \bar{u}(p')(\gamma^\mu\gamma^\alpha p_\alpha + \gamma^\alpha\gamma^\mu p'_\alpha)u(p) - 2m\bar{u}(p')\gamma^\mu u(p) = 0 \tag{41}$$

We can write the first term as,

$$\bar{u}(p')(\gamma^\mu\gamma^\alpha\frac{1}{2}((p_\alpha + p'_\alpha) + (p_\alpha - p'_\alpha)) + \gamma^\alpha\gamma^\mu\frac{1}{2}((p_\alpha + p'_\alpha) - (p_\alpha - p'_\alpha)))u(p) \tag{42}$$
$$= \frac{1}{2}(p_\alpha + p'_\alpha)\bar{u}(p')(\{\gamma^\mu, \gamma^\alpha\})u(p) + \frac{1}{2}(p_\alpha - p'_\alpha)\bar{u}(p')[\gamma^\mu, \gamma^\alpha]u(p) \tag{43}$$
$$= (p_\mu + p'_\mu)\bar{u}(p')u(p) - i\sigma^{\mu\alpha}(p_\alpha - p'_\alpha)\bar{u}(p')u(p) \tag{44}$$

Therefore, Gordon identity,

$$\bar{u}(p')\left[(p_\mu + p'_\mu) - i\sigma^{\mu\alpha}(p_\alpha - p'_\alpha)\right]u(p) - 2m\bar{u}(p')\gamma^\mu u(p) = 0 \tag{45}$$

Since the equation for the v spinor looks exactly like the u spinor case except for a relative negative sign on the mass parameter m, it is simple to write down the Gordon identity for v.

$$\bar{u}(p')\left[(p_\mu + p'_\mu) - i\sigma^{\mu\alpha}(p_\alpha - p'_\alpha)\right]u(p) + 2m\bar{u}(p')\gamma^\mu u(p) = 0 \tag{46}$$

4.5 - Completeness of spinors

Part (i) Ex 4.8 Lahiri and Pal

This is a special case of the Gordon identity. For completeness' sake, we do the calculation fully. We are given the following equations,

$$(\not{p} - m)u(p) = 0 \tag{47}$$
$$\bar{u}(p)(\not{p} - m) = 0 \tag{48}$$

Proceeding as suggested in the problem,

$$\bar{u}(p)\gamma^\mu(\not{p} - m)u(p) + \bar{u}(p)(\not{p} - m)\gamma^\mu u(p) = 0 \tag{49}$$
$$\Rightarrow \quad p_\alpha\bar{u}(p)(\gamma^\mu\gamma^\alpha + \gamma^\alpha\gamma^\mu)u(p) - 2m\bar{u}(p)\gamma^\mu u(p) = 0 \tag{50}$$
$$\Rightarrow \quad p^\mu\bar{u}(p)u(p) - m\,\bar{u}(p)\gamma^\mu u(p) = 0 \tag{51}$$

The same calculation with the v spinors yields,

$$p^\mu\bar{v}(p)v(p) + m\,\bar{v}(p)\gamma^\mu v(p) = 0 \tag{52}$$

If we put $\mu = 0$ in the u equation, we get,

$$
\begin{aligned}
& p^0 \bar{u}_s(p) u_r(p) - m\, \bar{u}_s(p) \gamma^0 u_r(p) = 0 && (53) \\
\Rightarrow \quad & E \bar{u}_s(p) u_r(p) - m\, u_s^\dagger(p) \gamma^0 \gamma^0 u_r(p) = 0 && (54) \\
\Rightarrow \quad & E \bar{u}_s(p) u_r(p) - m\, u_s^\dagger(p) u_r(p) = 0 && (55) \\
\Rightarrow \quad & E \bar{u}_s(p) u_r(p) - 2m\, E\, \delta_{rs} = 0 && (56) \\
\Rightarrow \quad & \bar{u}_s(p) u_r(p) = 2m\, \delta_{rs} && (57)
\end{aligned}
$$

$$
\begin{aligned}
& p^0 \bar{v}(p) v(p) + m\, \bar{v}(p) \gamma^0 v(p) = 0 && (58) \\
\Rightarrow \quad & \bar{v}_s(p) v_r(p) = -2m\, \delta_{rs} && (59)
\end{aligned}
$$

Part (ii) Ex 4.10 Lahiri and Pal

problem 4.3 / L&P Ex4.10 (7)

We first prove that $\sum_s u_s(p)\bar{u}_s(p) = \not{p}+m$

Recall that $\bar{u}_r(p)u_s(p) = -\bar{v}_r(p)v_s(p) = 2m\delta_{rs}$

and ~~then~~ the Dirac equations: $(\not{p}-m)u_s(p)=0$; $(\not{p}+m)v_s(p)=0$

We have
$$\sum_s u_s(p)\bar{u}_s(p)\,u_r(p)$$
$$= \sum_s u_s(p)(2m\delta_{rs})$$
$$= 2m\,u_r(p)$$
$$(\not{p}+m)u_r(p) = 2m\,u_r(p) \quad \text{using Dirac equation}$$

Also, $\sum_s u_s(p)\bar{u}_s(p)\,v_r(p) = 0$

and $(\not{p}+m)v_r(p) = 0$

Therefore, the two operators $\sum_s u_s(p)\bar{u}_s(p)$ and $\not{p}+m$ ~~agrees~~ agree with each other when acting on the four basis: $u_r(p)$, $v_r(p)$ $(r=1,2)$.

This means that $\sum_s u_s(p)\bar{u}_s(p) = \not{p}+m$.

We can prove $\sum_s v_s(p)\bar{v}_s(p) = \not{p}-m$ in a similar way:

$$\left.\begin{array}{l} \left\{\begin{array}{l} \sum_s v_s(p)\bar{v}_s(p)\,v_r(p) = -2m\,v_r(p) \\ (\not{p}-m)v_r(p) = -2m\,v_r(p) \end{array}\right. \\ \left\{\begin{array}{l} \sum_s v_s(p)\bar{v}_s(p)\,u_r(p) = 0 \\ (\not{p}-m)u_r(p) = 0 \end{array}\right. \end{array}\right\} \Rightarrow \sum_s v_s(p)\bar{v}_s(p) = \not{p}-m$$

17

4.6 - Helicity and chirality projection operators

problem 4.4 / L&P Ex 4.19 ⑧

$$\Sigma_p = \frac{\vec{\Sigma}\cdot\vec{p}}{p} = \frac{1}{p}\left(\sigma^{23}p_x + \sigma^{31}p_y + \sigma^{12}p_z\right)$$

The Dirac equation for massless fermion is

$$\not{p}\,\psi = 0$$

$$\Rightarrow \quad p\gamma_0 - p_x\gamma_1 - p_y\gamma_2 - p_z\gamma_3 = 0$$

$$\Rightarrow \Sigma_p\psi = \frac{1}{p}\left[\sigma^{23}p_x + \sigma^{31}p_y + \sigma^{12}p_z\right]\gamma^0\gamma^0\psi$$

$$= \frac{1}{p^2}\gamma^0\left[\sigma^{23}p_x + \sigma^{31}p_y + \sigma^{12}p_z\right]\left(p_x\gamma_1 + p_y\gamma_2 + p_z\gamma_3\right)\psi$$

(note that $\sigma^{23} = i\gamma^2\gamma^3$; $\sigma^{31} = i\gamma^3\gamma^1$; $\sigma^{12} = i\gamma^1\gamma^2$)

$$= \frac{i}{p^2}\gamma^0\Big[\gamma^1\gamma^2\gamma^3p_x^2 + \gamma^1\gamma^2\gamma^3p_y^2 + \gamma^1\gamma^2\gamma^3p_z^2 + p_xp_y\gamma^2\gamma^3\gamma^2 + p_xp_y\gamma^3\gamma^1\gamma^1$$

$$+ \gamma^2\gamma^3\gamma^3p_xp_z + \gamma^1\gamma^2\gamma^1p_xp_z + \gamma^3\gamma^1\gamma^3p_yp_z + \gamma^1\gamma^2\gamma^2p_yp_z\Big]\psi$$

$$= \frac{i}{p^2}\gamma^0\gamma^1\gamma^2\gamma^3p^2\psi$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3\psi$$

$$= \gamma_5\psi$$

Therefore: $\frac{1}{2}(1\pm\Sigma_p)\psi = \frac{1}{2}(1\pm\gamma_5)\psi$ for massless fermion. □