

## Homework 3 Solutions

### 3.1: $U(1)$ symmetry for complex scalar

Solutions to HWK 3  
PHYS 624

(1)

#### Problem 3.1

$$\begin{aligned}
 \underline{3.1.1} \quad \mathcal{L} &= \sum_{A=1}^2 \left[ \frac{1}{2} (\partial^\mu \phi_A) (\partial_\mu \phi_A) - \frac{1}{2} m_A^2 \phi_A^2 \right] \\
 &= \frac{1}{2} [\partial^\mu (\phi_1 - i\phi_2)] [\partial_\mu (\phi_1 + i\phi_2)] - \frac{1}{2} \left( \frac{m_1^2 + m_2^2}{2} \right) (\phi_1^2 + \phi_2^2) \\
 &\quad + \frac{1}{2} \left( \frac{m_2^2 - m_1^2}{2} \right) (\phi_1^2 - \phi_2^2) \\
 &= \partial^\mu \phi^\dagger \partial_\mu \phi - \frac{1}{2} (m_1^2 + m_2^2) \phi^\dagger \phi + \frac{1}{4} (m_2^2 - m_1^2) (\phi^2 + \phi^{\dagger 2})
 \end{aligned}$$

Under the transformation  $\phi \rightarrow e^{-i\theta} \phi$

The first two terms  $\partial^\mu \phi^\dagger \partial_\mu \phi - \frac{1}{2} (m_1^2 + m_2^2) \phi^\dagger \phi$  is invariant.

But the term  $\frac{1}{4} (m_2^2 - m_1^2) (\phi^2 + \phi^{\dagger 2}) \rightarrow \frac{1}{4} (m_2^2 - m_1^2) [e^{-2i\theta} \phi^2 + e^{2i\theta} \phi^{\dagger 2}]$ , which is not invariant!

Therefore, there is no motivation to use complex scalar field when  $m_1 \neq m_2$ .  $\square$

#### 3.1.2 / Ex 3.7 of L&P

$$Q = q \int d^3p [a^\dagger(p) a(p) - \bar{a}^\dagger(p) \bar{a}(p)]$$

$$\text{where } a(p) = \frac{1}{\sqrt{2}} (a_1(p) + i a_2(p)) ; \quad \bar{a}(p) = \frac{1}{\sqrt{2}} (a_1(p) - i a_2(p))$$

$$\Rightarrow Q = q \int d^3p \left\{ \frac{1}{2} [a_1^\dagger(p) - i a_2^\dagger(p)] [a_1(p) + i a_2(p)] - \frac{1}{2} [a_1^\dagger(p) + i a_2^\dagger(p)] [a_1(p) - i a_2(p)] \right\}$$

$$\begin{aligned}
 \Rightarrow Q &= i \int d^3p \left\{ \frac{1}{2} [a_1^\dagger(p) a_1(p) + i a_1^\dagger(p) a_2(p) - i a_2^\dagger(p) a_1(p) + a_2^\dagger(p) a_2(p)] \right. \\
 &\quad \left. - \frac{1}{2} [a_1^\dagger(p) a_1(p) - i a_1^\dagger(p) a_2(p) + i a_2^\dagger(p) a_1(p) + a_2^\dagger(p) a_2(p)] \right\} \\
 &= i \int d^3p (i a_1^\dagger(p) a_2(p) - i a_2^\dagger(p) a_1(p))
 \end{aligned}$$

We see there are cross term between  $a_1^\dagger(p)$  &  $a_2(p)$ , etc.

which means that  $Q$  cannot be written as number operators of  $\phi_1$  &  $\phi_2$ .  $\square$

### problem 3.2

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{3/2} \sqrt{2E_p}} [a(p) e^{-ip \cdot x} + \bar{a}^\dagger(p) e^{ip \cdot x}]$$

$$\phi^\dagger(x) = \int \frac{d^3p}{(2\pi)^{3/2} \sqrt{2\bar{E}_p}} [\bar{a}(p) e^{-ip \cdot x} + a^\dagger(p) e^{ip \cdot x}]$$

where  $E_p = \sqrt{p^2 + m^2}$ ;  $\bar{E}_p = \sqrt{p^2 + \bar{m}^2}$  with  $m, \bar{m}$  the masses for particle & anti particle respectively.

The commutator

$$\begin{aligned}
 [\phi(x), \phi^\dagger(y)] &= \int \frac{d^3p d^3p'}{(2\pi)^3 \sqrt{4E_p \bar{E}_{p'}}} \left[ [a(p), a^\dagger(p')] e^{-ip \cdot x + ip' \cdot y} + [\bar{a}^\dagger(p), \bar{a}(p')] e^{ip \cdot x - ip' \cdot y} \right] \\
 &= \int \frac{d^3p}{(2\pi)^3 \sqrt{4E_p \bar{E}_p}} \left\{ e^{i\vec{p} \cdot (\vec{x} - \vec{y}) - iE_p t + i\bar{E}_p t'} - e^{-i\vec{p} \cdot (\vec{x} - \vec{y}) + iE_p t - i\bar{E}_p t'} \right\}
 \end{aligned}$$

Causality requires that the commutator vanishes if  $x, y$  are spacelike separated.

### 3.2: Two complex scalars

The Lagrangian for two complex scalar fields is given by,

$$\mathcal{L} = \partial_\mu \phi_1^* \partial^\mu \phi_1 - m^2 \phi_1^* \phi_1 + \partial_\mu \phi_2^* \partial^\mu \phi_2 - m^2 \phi_2^* \phi_2 \quad (1)$$

This can be written in a more compact (and physically intuitive) form, with the following identification,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (2)$$

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi \quad (3)$$

We start with the symmetries which are obvious,

$$\Phi \rightarrow e^{i\alpha} \Phi \quad (4)$$

$$\Rightarrow \delta\Phi = i\alpha\Phi \quad (5)$$

This is just a generalization of the  $U(1)$  symmetry in the case of a single complex scalar field. The corresponding charge is

$$Q = \int d^3x \frac{\delta\mathcal{L}}{\delta(\partial_0\Phi)}(i\Phi) + (-i\Phi^\dagger) \frac{\delta\mathcal{L}}{\delta(\partial_0\Phi^\dagger)} \quad (6)$$

$$= i \int d^3x ((\partial_0\Phi^\dagger)\Phi - \Phi^\dagger(\partial_0\Phi)) \quad (7)$$

$$= i \int d^3x (\pi_1\phi_1 - \phi_1^*\pi_1^*) + (\pi_2\phi_2 - \phi_2^*\pi_2^*) \quad (8)$$

Remember that the charge is only defined up to an overall constant.

There is also an  $SU(2)$  symmetry, since  $\Phi^\dagger\Phi$  is invariant under unitary transformations.

$$\Phi \rightarrow e^{\frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}}\Phi \quad (9)$$

$$\Rightarrow \delta\Phi = \frac{i}{2}\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}\Phi \quad (10)$$

The conserved charge for this symmetry is,

$$Q^j = \int d^3x \frac{\delta\mathcal{L}}{\delta(\partial_0\Phi)}(\frac{i}{2}\sigma^j\Phi) + (-\frac{i}{2}\sigma^j\Phi^\dagger) \frac{\delta\mathcal{L}}{\delta(\partial_0\Phi^\dagger)} \quad (11)$$

$$= \frac{i}{2} \int d^3x ((\partial_0\Phi^\dagger)\sigma^j\Phi - \Phi^\dagger\sigma^j(\partial_0\Phi)) \quad (12)$$

$$= \frac{i}{2} \int d^3x (\pi_a(\sigma^j)_{ab}\phi_b - \phi_a^*(\sigma^j)_{ab}\pi_b^*) \quad (13)$$

Again, the overall constant is not important. In transforming  $\Phi^\dagger$ , we use the fact that the Pauli matrices are hermitian.

The commutator,

$$[Q^i, Q^j] = -\frac{1}{4} \int d^3x d^3x' [\pi_a(\sigma^i)_{ab} \phi_b - \phi_a^*(\sigma^i)_{ab} \pi_b^*, \pi'_c(\sigma^j)_{cd} \phi'_d - \phi'^*_c(\sigma^j)_{cd} \pi'^*_d] \quad (14)$$

where  $\phi' \rightarrow \phi(x')$  etc. Consider one of the four terms

$$[\pi_a(\sigma^i)_{ab} \phi_b, \pi'_c(\sigma^j)_{cd} \phi'_d] = \pi_a(\sigma^i)_{ab} \phi_b \pi'_c(\sigma^j)_{cd} \phi'_d - \pi'_c(\sigma^j)_{cd} \phi'_d \pi_a(\sigma^i)_{ab} \phi_b \quad (15)$$

$$= \pi_a (\pi'_c \phi'_d + i\delta^3(\mathbf{x} - \mathbf{x}') \delta_{bc}) \phi'_d (\sigma^i)_{ab} (\sigma^j)_{cd} \quad (16)$$

$$- \pi'_c (\pi_a \phi'_d + i\delta^3(\mathbf{x} - \mathbf{x}') \delta_{ad}) \phi_b (\sigma^i)_{ab} (\sigma^j)_{cd} \quad (17)$$

$$= (\pi_a \pi'_c \phi_b \phi'_d - \pi'_c \pi_a \phi'_d \phi_b) (\sigma^i)_{ab} (\sigma^j)_{cd} \quad (18)$$

$$- i\delta^3(\mathbf{x} - \mathbf{x}') (\pi_a \phi'_d (\sigma^i \sigma^j)_{ad} - \pi'_c \phi_b (\sigma^j \sigma^i)_{cb}) \quad (19)$$

Now, we know that  $[\pi, \pi'] = [\phi, \phi'] = 0$ . This means the first term goes to zero. In the second term, we can switch the dummy indices  $\{a, b\} \leftrightarrow \{c, d\}$  since they are summed over, and can also switch the primes, because  $x$  and  $x'$  are integrated over. This yields,

$$[\pi_a(\sigma^i)_{ab} \phi_b, \pi'_c(\sigma^j)_{cd} \phi'_d] = -i\delta^3(\mathbf{x} - \mathbf{x}') \pi_a \phi'_d [\sigma^i, \sigma^j]_{ad} \quad (20)$$

$$= -i\delta^3(\mathbf{x} - \mathbf{x}') \pi_a (2i\epsilon^{ijk} \sigma_k)_{ad} \phi'_d \quad (21)$$

Taking into account the fact that  $[\phi, \phi^*] = [\pi, \pi^*] = 0$ , the cross-terms will not contribute. Hence,

$$[Q^i, Q^j] = -\frac{1}{2} \epsilon^{ijk} \int d^3x (\pi_a (\sigma_k)_{ab} \phi_b - \pi_a^* (\sigma_k)_{ab} \phi_b^*) \quad (22)$$

$$= i\epsilon^{ijk} Q_k \quad (23)$$

which is the  $SU(2)$  algebra.

### Problem 3.3: Causality

(i)

The expansion of  $\phi(x)$ ,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}} \sqrt{2E_p}} [a(p) e^{-ip \cdot x} + b^\dagger(p) e^{ip \cdot x}] \quad (24)$$

The commutator is,

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}} \sqrt{2E_p}} \frac{d^3p'}{(2\pi)^{\frac{3}{2}} \sqrt{2E_{p'}}} \left[ [a(p), a^\dagger(p')] e^{-ip \cdot x + ip' \cdot y} + [b^\dagger(p), b(p')] e^{ip \cdot x - ip' \cdot y} \right] \quad (25)$$

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3p}{(2\pi)^3 2E_p} [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] \quad (26)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ e^{i\vec{p} \cdot (\vec{x} - \vec{y})} e^{-iE_p(t-t')} - e^{-i\vec{p} \cdot (\vec{x} - \vec{y})} e^{iE_p(t-t')} \right] \quad (27)$$

(ii)

At equal times,

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3p}{(2\pi)^3 2E_p} [e^{i\vec{p} \cdot (\vec{x} - \vec{y})} - e^{-i\vec{p} \cdot (\vec{x} - \vec{y})}] \quad (28)$$

which is an integral odd in  $\vec{p}$ , and hence is zero.

(iii)

$$[\phi(x), \phi^\dagger(y)] = \int \frac{d^3p}{(2\pi)^3 2E_p} [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] \quad (29)$$

The integral can be rewritten,

$$\int \frac{d^3p}{(2\pi)^3 2E_p} [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] = \int \frac{d^4p}{(2\pi)^4} (2\pi) \delta(p^2 - m^2) [e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)}] \Big|_{p_0 > 0} \quad (30)$$

in which form, it is clear that each term separately is Lorentz invariant.

(iv)

If  $x$  and  $y$  are space-like separated, it is always possible to find a reference frame where the time-separation is zero. In that frame, it follows from above that the commutator vanishes. Since it is Lorentz-invariant, it follows that it vanishes for space-like interval.

(v)

Clearly, for time-like separated events we cannot find a frame where the time-difference is zero. Thus, the above argument does not go through.

(vi)

Since the commutator is a  $c$ -number, its matrix element in the vacuum state is the same as itself,

$$[\phi(x), \phi^\dagger(y)] = \langle 0 | [\phi(x), \phi^\dagger(y)] | 0 \rangle \quad (31)$$

$$= \langle 0 | \phi(x) \phi^\dagger(y) | 0 \rangle - \langle 0 | \phi^\dagger(y) \phi(x) | 0 \rangle \quad (32)$$

Since  $\phi(x) | 0 \rangle$  stands for a particle at position  $x$ , and  $\phi^\dagger(y) | 0 \rangle$  denotes an antiparticle at position  $y$ . Similarly  $\langle 0 | \phi(x)$  is an anti-particle at  $x$ . Therefore, the first term represents the anti-particle propagation from  $y$  to  $x$ , and the second represents particle propagation from  $x$  to  $y$ .

Therefore, we see for space-like separated  $x$  and  $y$ , particle propagation from  $x$  to  $y$  cancels anti-particle propagation  $y$  to  $x$ . Therefore, in order to ensure causality, we require the presence of an anti-particle.

### Problem 3.4: Propagator and Green's function

The Feynman propagator is,

$$\Delta_F(x - x') = -i \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \theta(t - t') e^{-ip \cdot (x - x')} + \theta(t' - t) e^{ip \cdot (x - x')} \right] \quad (33)$$

$$\begin{aligned} \partial_t^2 \Delta_F(x - x') = & -i \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \delta'(t - t') e^{-ip \cdot (x - x')} - \delta'(t' - t) e^{ip \cdot (x - x')} \right. \\ & + 2\delta'(t - t')(-iE_p) e^{ip \cdot (x - x')} - 2\delta'(t' - t)(iE_p) e^{-ip \cdot (x - x')} \\ & \left. + \theta(t - t')(-E_p^2) e^{-ip \cdot (x - x')} + \theta(t' - t)(-E_p^2) e^{ip \cdot (x - x')} \right] \end{aligned} \quad (34)$$

Using integration by parts, we can make use of the following identity,

$$\delta'(x) f(x) \rightarrow \delta(x) f'(x) \quad (35)$$

Thus,

$$\partial_t^2 \Delta_F(x - x') = - \int \frac{d^3p}{(2\pi)^3 2E_p} \delta(t - t') e^{-ip \cdot (x - x')} \quad (36)$$

$$+ i \int \frac{d^3p}{(2\pi)^3 2E_p} E_p^2 \left[ \theta(t - t') e^{-ip \cdot (x - x')} + \theta(t' - t) e^{ip \cdot (x - x')} \right] \quad (37)$$

$$= -\delta^4(x - x') + i \int \frac{d^3p}{(2\pi)^3 2E_p} E_p^2 \left[ \theta(t - t') e^{-ip \cdot (x - x')} + \theta(t' - t) e^{ip \cdot (x - x')} \right] \quad (38)$$

The space derivatives are much more straight-forward. Thus,

$$(\square + m^2) \Delta_F(x - x') = -\delta^4(x - x') \quad (39)$$

$$+ i \int \frac{d^3p}{(2\pi)^3 2E_p} (E_p^2 - \vec{p}^2 - m^2) \left[ \theta(t - t') e^{-ip \cdot (x - x')} + \theta(t' - t) e^{ip \cdot (x - x')} \right] \quad (40)$$

$$= -\delta^4(x - x') \quad (41)$$

Thus, the Feynman propagator is the Green's function of the Klein-Gordon equation.

### Problem 3.5: Propagator for complex scalar field

The time-ordered product,

$$\langle 0 | \mathcal{T} [\phi(x) \phi^\dagger(x')] | 0 \rangle = \theta(t - t') \langle 0 | \phi(x) \phi^\dagger(x') | 0 \rangle + \theta(t' - t) \langle 0 | \phi^\dagger(x') \phi(x) | 0 \rangle \quad (42)$$

$$\begin{aligned} &= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \theta(t - t') \langle 0 | a(p) a^\dagger(p') | 0 \rangle e^{-ip \cdot (x - x')} \right. \\ &\quad \left. + \theta(t' - t) \langle 0 | b(p) b^\dagger(p') | 0 \rangle e^{ip \cdot (x - x')} \right] \end{aligned} \quad (43)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \theta(t - t') e^{-ip \cdot (x - x')} + \theta(t' - t) e^{ip \cdot (x - x')} \right] \quad (44)$$

$$= i \Delta_F(x - x') \quad (45)$$

a)

$$i\Delta_F(x - x') = \theta(t - t') \langle 0 | \phi(x) \phi^\dagger(x') | 0 \rangle + \theta(t' - t) \langle 0 | \phi^\dagger(x') \phi(x) | 0 \rangle \quad (46)$$

In analogy with Problem 3.3. part vi, we see that if  $t > t'$ , then the time-ordered correlation function propagates an anti-particle from  $t'$  to  $t$ . If  $t < t'$  it represents the propagation of a particle from  $t$  to  $t'$ .

b)

If we start with,

$$\langle 0 | \mathcal{T} [\phi(x') \phi^\dagger(x)] | 0 \rangle = \theta(t' - t) \langle 0 | \phi(x') \phi^\dagger(x) | 0 \rangle + \theta(t - t') \langle 0 | \phi^\dagger(x) \phi(x') | 0 \rangle \quad (47)$$

$$= \int \frac{d^3p \, d^3p'}{(2\pi)^3 2E_p} \left[ \theta(t' - t) \langle 0 | a(p) a^\dagger(p') | 0 \rangle e^{-ip \cdot (x' - x)} + \theta(t - t') \langle 0 | b(p) b^\dagger(p') | 0 \rangle e^{ip \cdot (x' - x)} \right] \quad (48)$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_p} \left[ \theta(t - t') e^{-ip \cdot (x - x')} + \theta(t' - t) e^{ip \cdot (x - x')} \right] \quad (49)$$

$$= i\Delta_F(x - x') \quad (50)$$

In other words,

$$\Delta_F(x - x') = \Delta_F(x' - x) \quad (51)$$

Remember that we have not said if  $x$  and  $x'$  are space-like or time-like separated. The above equations hold true for any pair  $x$  and  $x'$ . Let us pick a particular ordering,  $t > t'$ . Then, in equation (46), only the first term contributes. We would then conclude that the propagator represents an anti-particle propagating from  $t'$  to  $t$  (which is comfortably the forward direction in time).

On the other hand, we show in part (b) that there is another way to write the propagator. In equation (47), the term that contributes for  $t > t'$  is,

$$\theta(t - t') \langle 0 | \phi^\dagger(x) \phi(x') | 0 \rangle \quad (52)$$

which represents the propagation of a particle from  $t'$  to  $t$ .

Therefore, the Feynman propagator can be understood to propagate either particles or anti-particles from one spacetime point to another.