Homework 2 Solutions

Problem 2.1: Toy Model for radiation field

PHYS 624, Solutions to HWK 2 ①

2.1 / Ex. 2.7 of L&P

• zeroth step.

The Lagrangian $\mathcal{L} = \left(\frac{\partial u}{\partial t}\right)^2 - c^2\left(\frac{\partial u}{\partial x}\right)^2$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial(\partial_0 u)} = 2\frac{\partial u}{\partial t} \; ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_x u)} = -2c^2 \frac{\partial u}{\partial x}$$

The E-L equation is

$$\partial_\mu\left(\frac{\partial}{\partial(\partial_\mu u)}\mathcal{L}\right) = 0$$

This leads to $\frac{\partial^2}{\partial t^2} u - c^2 \frac{\partial^2 u}{\partial x^2} = 0$, which is just the wave equation !

• a) First, we need to write $\mathcal{L}$ in terms of q_k and $\dot{q}_k$

$$\mathcal{L} = \int_0^\ell dx \left[\left(\sum_{k=1}^{\infty} \dot{q}_k \sin\frac{w_k x}{c}\right)^2 - \left(\sum_{k=1}^{\infty} q_k w_k \cos\frac{w_k x}{c}\right)^2\right]$$

Use the following identities:

$$\int_0^\ell dx \, \sin\frac{w_k x}{c} \sin\frac{w_j x}{c} \, dx = \frac{\ell}{2}\delta_{kj} = \int_0^\ell dx \, \cos\frac{w_k x}{c} \cos\frac{w_j x}{c} \, dx$$

$\left(\text{note } w_k = \frac{k\pi c}{\ell}\right)$

(2)

we arrive at

$$\mathcal{L} = \sum_{k=1}^{\infty} \frac{\ell}{2}(\dot{q}_k^2 - \omega_k^2 q_k^2)$$

Then we can get

$$P_k = \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \ell \dot{q}_k$$

· b) The Hamiltonian is

$$H = \sum_k (P_k \dot{q}_k - \mathcal{L})$$
$$= \frac{1}{2}\sum_k \left(\frac{P_k^2}{\ell} + \ell \omega_k^2 q_k^2\right)$$

· c) $q_k = \sqrt{\frac{\hbar}{2\ell\omega_k}}\left[a_k e^{-i\omega_k t} + a_k^+ e^{i\omega_k t}\right]$

$$\Rightarrow P_k = \ell \dot{q}_k = i\sqrt{\frac{\hbar \ell \omega_k}{2}}\left[a_k^+ e^{i\omega_k t} - a_k e^{-i\omega_k t}\right]$$

We can invert these two equations to get

$$a_k^+ = \sqrt{\frac{\ell \omega_k}{2\hbar}}\; e^{i\omega_k t}\left[q_k - \frac{i}{\omega_k \ell} P_k\right]$$
$$a_k = \sqrt{\frac{\ell \omega_k}{2\hbar}}\; e^{-i\omega_k t}\left[q_k + \frac{i}{\omega_k \ell} P_k\right]$$

Then we can get the commutation relations

$$[a_k, a_j^+] = \frac{\ell\sqrt{\omega_k \omega_j}}{2\hbar} e^{-i(\omega_k - \omega_j)t}\left[\frac{+\hbar}{\omega_k \ell}\delta_{kj} + \frac{\hbar}{\omega_j \ell}\delta_{kj}\right] = \delta_{kj}$$

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$$[a_k, a_j] = -\frac{\ell\sqrt{\omega_k\omega_j}}{2\hbar} e^{-i(\omega_k+\omega_j)t}\left[\frac{\hbar}{\omega_k \ell}\delta_{kj} - \frac{\hbar}{\omega_j \ell}\delta_{kj}\right] = 0$$

similarly $[a_k^+, a_j^+] = 0$

• d) The Hamiltonian is

$$H = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{P_k^2}{\ell} + \ell\,\omega_k^2 q_k^2\right)$$

$$\ell\sum_k \omega_k^2 q_k^2 = \sum_k \left(\frac{\hbar\omega_k}{2}\right)\left(a_k e^{-i\omega_k t} + a_k^+ e^{i\omega_k t}\right)^2$$

$$= \sum_K \frac{\hbar\omega_k}{2}\left(a_k^2 e^{-2i\omega_k t} + a_k^{+2} e^{2i\omega_k t} + a_k a_k^+ + a_k^+ a_k\right)$$

$$\sum_k \frac{P_k^2}{\ell} = \sum_k\left(-\frac{\hbar\omega_k}{2}\right)\left(a_k^{+2} e^{2i\omega_k t} + a_k^2 e^{-2i\omega_k t} - a_k a_k^+ - a_k^+ a_k\right)$$

$$\Rightarrow H = \sum_k \frac{\hbar\omega_k}{2}\left(a_k^+ a_k + a_k a_k^+\right) = \sum_k \hbar\omega_k\left(a_k^+ a_k + \frac{1}{2}\right)$$

□

$\boxed{2.2}$ / Ex. 3.4 of L&P   For simplicity, we can set $t=0$.

$$\phi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left(a(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^+(p)\,e^{i\vec{p}\cdot\vec{x}}\right)$$

$$\Pi(x) = \int d^3p\; i\sqrt{\frac{E_p}{2(2\pi)^3}}\left(-a(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^+(p)\,e^{i\vec{p}\cdot\vec{x}}\right)$$

$$\tilde{\phi}(k) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}\phi(x) = \int \frac{d^3x\, d^3p}{(2\pi)^3\sqrt{2E_p}}\left[a(p)\,e^{i(\vec{k}-\vec{p})\cdot\vec{x}} + a^+(p)\,e^{i(\vec{k}+\vec{p})\cdot\vec{x}}\right]$$

$$= \sqrt{\frac{1}{2E_k}}\cdot\left(a(k) + a^+(-k)\right)$$

Problem 2.2: Commutation relations for $a, a^\dagger$

Please ignore the text above the line.

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$$[a_k, a_j] = -\frac{\ell\sqrt{\omega_k\omega_j}}{2\hbar} e^{-i(\omega_k+\omega_j)t}\left[\frac{\hbar}{\omega_k \ell}\delta_{kj} - \frac{\hbar}{\omega_j \ell}\delta_{kj}\right] = 0$$

similarly $[a_k^+, a_j^+] = 0$

• d) The Hamiltonian is

$$H = \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{P_k^2}{\ell} + \ell\,\omega_k^2 q_k^2\right)$$

$$\ell\sum_k \omega_k^2 q_k^2 = \sum_k\left(\frac{\hbar\omega_k}{2}\right)\left(a_k e^{-i\omega_k t} + a_k^+ e^{i\omega_k t}\right)^2$$

$$= \sum_k \frac{\hbar\omega_k}{2}\left(a_k^2 e^{-2i\omega_k t} + a_k^{+2} e^{2i\omega_k t} + a_k a_k^+ + a_k^+ a_k\right)$$

$$\sum_k \frac{P_k^2}{\ell} = \sum_k\left(-\frac{\hbar\omega_k}{2}\right)\left(a_k^{+2} e^{2i\omega_k t} + a_k^2 e^{-2i\omega_k t} - a_k a_k^+ - a_k^+ a_k\right)$$

$$\Rightarrow H = \sum_k \frac{\hbar\omega_k}{2}\left(a_k^+ a_k + a_k a_k^+\right) = \sum_k \hbar\omega_k\left(a_k^+ a_k + \frac{1}{2}\right)$$

□

$\boxed{2.2}$ / Ex. 3.4 of L&P — For simplicity, we can set $t=0$.

$$\phi(x) = \int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left(a(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^+(p)\,e^{i\vec{p}\cdot\vec{x}}\right)$$

$$\Pi(x) = \int d^3p\; i\sqrt{\frac{E_p}{2(2\pi)^3}}\left(-a(p)\,e^{-i\vec{p}\cdot\vec{x}} + a^+(p)\,e^{i\vec{p}\cdot\vec{x}}\right)$$

$$\tilde{\phi}(k) = \int\frac{d^3x}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}\phi(x) = \int\frac{d^3x\,d^3p}{(2\pi)^3\sqrt{2E_p}}\left[a(p)\,e^{i(\vec{k}-\vec{p})\cdot\vec{x}} + a^+(p)\,e^{i(\vec{k}+\vec{p})\cdot\vec{x}}\right]$$

$$= \sqrt{\frac{1}{2E_k}}\cdot\left(a(k) + a^+(-k)\right)$$

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$$\tilde{\Pi}(k) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \Pi(x) = \int \frac{d^3x\, d^3p}{(2\pi)^3} \, i\sqrt{\frac{E_p}{2}} \left[-a(p) e^{i(\vec{k}-\vec{p})\cdot\vec{x}} + a^\dagger(p) e^{i(\vec{k}\cdot\vec{x}+\vec{p}\cdot\vec{x})} \right]$$

$$= i\sqrt{\frac{E_k}{2}} \left[-a(k) + a^\dagger(-k) \right]$$

We can invert these two equations to get

$$a(k) = \frac{1}{2} \left[\sqrt{2E_k}\, \tilde{\phi}(k) + i\sqrt{\frac{2}{E_k}}\, \tilde{\Pi}(k) \right]$$

$$= \frac{1}{2} \int \frac{d^3x}{(2\pi)^{3/2}} \cdot \left[\sqrt{2E_k}\, \phi(x) + i\sqrt{\frac{2}{E_k}}\, \Pi(x) \right] e^{i\vec{k}\cdot\vec{x}}$$

$$a^\dagger(k) = \frac{1}{2} \left[\sqrt{2E_k}\, \tilde{\phi}(-k) - i\sqrt{\frac{2}{E_k}}\, \tilde{\Pi}(-k) \right]$$

$$= \frac{1}{2} \int \frac{d^3x}{(2\pi)^{3/2}} \left[\sqrt{2E_k}\, \phi(x) - i\sqrt{\frac{2}{E_k}}\, \Pi(x) \right] e^{-i\vec{k}\cdot\vec{x}}$$

Now we can get the commutation relations:

$$\left[a(k), a^\dagger(p) \right] = \frac{1}{4} \int \frac{d^3x\, d^3y}{(2\pi)^3} \cdot 2\sqrt{2E_k} \cdot \sqrt{\frac{2}{E_p}} \cdot \delta^3(\vec{x}-\vec{y})\, e^{i\vec{k}\cdot\vec{x} - i\vec{p}\cdot\vec{y}}$$

$$= \int \frac{d^3x}{(2\pi)^3} \sqrt{\frac{E_k}{E_p}} \cdot e^{i(\vec{k}-\vec{p})\cdot\vec{x}}$$

$$= \delta^3(\vec{k}-\vec{p})$$

$$\left[a(k), a(p) \right] = \frac{1}{4} \int \frac{d^3x\, d^3y}{(2\pi)^3} \cdot \left[-\sqrt{2E_k}\sqrt{\frac{2}{E_p}} \cdot \delta^3(\vec{x}-\vec{y}) + \sqrt{\frac{2}{E_k}}\sqrt{2E_p}\, \delta^3(\vec{x}-\vec{y}) \right] e^{i\vec{k}\cdot\vec{x} - i\vec{p}\cdot\vec{y}}$$

$$= 0$$

similarly $\left[a^\dagger(k), a^\dagger(p) \right] = 0$ □.

Problem 2.3: Momentum Operator

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$\boxed{2.3}$ 5

The 3-momentum operator is

$$\mathbb{P}_i = -\int d^3x \, \dot\phi \, \partial_i \phi$$

we have

$$\dot\phi = \int \frac{d^3p}{(2\pi)^{3/2}} \, i\sqrt{\frac{E_p}{2}} \left[-a(p) \, e^{-ip\cdot x} + a^\dagger(p) \, e^{ip\cdot x} \right]$$

$$\partial_i \phi = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E_k}} \, i \left[a(k) e^{-ik\cdot x} - a^\dagger(k) \, e^{ik\cdot x} \right] k_i$$

$$\Rightarrow \mathbb{P}_i = -\int \frac{d^3x \, d^3k \, d^3p}{(2\pi)^3} \left(-\frac{1}{2}\right) \sqrt{\frac{E_p}{E_k}} \left[-a(p) a(k) e^{-i(p+k)\cdot x} - a^\dagger(p) a^\dagger(k) \, e^{i(p+k)\cdot x} \right.$$
$$\left. + \, a(p) a^\dagger(k) e^{i(k-p)\cdot x} + a^\dagger(p) \, a(k) \, e^{i(p-k)\cdot x} \right] k_i$$

$$= \frac{1}{2} \int d^3k \left[-a(-k) \, a(k) \, e^{-i2E_k t} - a^\dagger(-k) \, a^\dagger(k) \, e^{i2E_k t} \right.$$
$$\left. + \, a(k) a^\dagger(k) + a^\dagger(k) \, a(k) \right] k_i \qquad (*)$$

Since

$$\int d^3k \; a(-k) a(k) e^{-i2E_k t} k_i \;\underset{=}{\overset{k\to -k}{=}}\; -\int d^3k \; a(k) \, a(-k) e^{-2iE_k t} k_i$$

and $[a(k), a(-k)] = 0$

Therefore the first two terms in $(*)$ vanishes.

$$\Rightarrow \mathbb{P}_i = \frac{1}{2} \int d^3k \, \{ a(k) a^\dagger(k) + a^\dagger(k) a(k) \} k_i = \int d^3k \left[a^\dagger(k) a(k) + \tfrac{1}{2}\delta(0) \right] k_i$$

Again, since $\int d^3k \, k_i \overset{k\to -k}{=} -\int d^3k \, k_i \Rightarrow \int d^3k \, k_i = 0$

$$\Rightarrow \mathbb{P}_i = \int d^3k \; a^\dagger(k) a(k) \; k_i$$

We see that normal ordering is not required. $\square$

(ii) Exercise 3.5 b) of Lahiri and Pal

We know that

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right] \tag{1}$$

and

$$P_\mu = \int d^3k \; a^\dagger(k)a(k)k_\mu \tag{2}$$

Therefore,

$$\begin{aligned}
[\phi, P_\mu] &= \left[\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}, \int d^3k \; a^\dagger(k)a(k)k_\mu\right] &(3)\\
&= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \int d^3k \; k_\mu \left(e^{-ip\cdot x}\left[a(p), a^\dagger(k)a(k)\right] + e^{ip\cdot x}\left[a^\dagger(p), a^\dagger(k)a(k)\right]\right) &(4)\\
&= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \int d^3k \; k_\mu \left(e^{-ip\cdot x}\left[a(p), a^\dagger(k)\right]a(k) + e^{ip\cdot x}a^\dagger(k)\left[a^\dagger(p), a(k)\right]\right) &(5)\\
&= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \int d^3k \; k_\mu \left(e^{-ip\cdot x}\delta^3(\vec{p}-\vec{k})a(k) - e^{ip\cdot x}\delta^3(\vec{p}-\vec{k})a^\dagger(k)\right) &(6)\\
&= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} p_\mu \left(e^{-ip\cdot x}a(p) - e^{ip\cdot x}a^\dagger(p)\right) &(7)\\
&= \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} i\partial_\mu \left(e^{-ip\cdot x}a(p) + e^{ip\cdot x}a^\dagger(p)\right) &(8)\\
&= i\partial_\mu\phi &(9)
\end{aligned}$$

Problem 2.4: More examples

(i) Lahiri and Pal - Exercise 3.3

As before,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} \left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}\right] \tag{10}$$

$$\partial_i\phi(y) = \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}} - ip_i \left[a(p)e^{-ip\cdot y} - a^\dagger(p)e^{ip\cdot y}\right] \tag{11}$$

Assuming $x^0 = y^0 = t$, we want to calculate,

$$[\phi(x), \partial_i\phi(y)] = -i\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}}\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_k}} k_i \left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{ip\cdot x}, a(k)e^{-ik\cdot y} - a^\dagger(k)e^{ik\cdot y}\right] \tag{12}$$

$$= -i\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}}\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_k}} k_i \left(\left[a(p)e^{-ip\cdot x}, -a^\dagger(k)e^{ik\cdot y}\right] + \left[a^\dagger(p)e^{ip\cdot x}, a(k)e^{-ik\cdot y}\right]\right) \tag{13}$$

$$= -i\int \frac{d^3p}{(2\pi)^{\frac{3}{2}}\sqrt{2E_p}}\int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_k}} k_i \left(-\delta^3(\vec{p}-\vec{k})e^{-i(p\cdot x - k\cdot y)} - \delta^3(\vec{p}-\vec{k})e^{i(p\cdot x - k\cdot y)}\right) \tag{14}$$

$$= i\int \frac{d^3p}{(2\pi)^3 2E_p} p_i \left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right) \tag{15}$$

$$= i\int \frac{p^2\, dp\, d\Omega}{(2\pi)^3 2E_p} p_i \left(e^{-ip\cdot(x-y)} + e^{ip\cdot(x-y)}\right) \tag{16}$$

All factors in the integrand except p_i are even in p, therefore, the integral itself is odd. Since the integration is from $-\infty$ to $+\infty$, an odd integrand will integrate to 0.

(ii) Lahiri and Pal - Exercise 3.6

The number operator is defined as,

$$\mathcal{N} = \int d^3p\; a^\dagger(p)a(p) \tag{17}$$

in analogy with usual quantum mechanics. The commutation relations of this operator are easy to derive,

$$[\mathcal{N}, a^\dagger(k)] = \int d^3p\; \left[a^\dagger(p)a(p), a^\dagger(k)\right] \tag{18}$$

$$= \int d^3p\; a^\dagger(p)\left[a(p), a^\dagger(k)\right] \tag{19}$$

$$= \int d^3p\; a^\dagger(p)\,\delta^3(\vec{p}-\vec{k}) = a^\dagger(k) \tag{20}$$

$$[\mathcal{N}, a(k)] = \int d^3p\; \left[a^\dagger(p)a(p), a(k)\right] \tag{21}$$

$$= \int d^3p\; \left[a^\dagger(p), a^\dagger(k)\right] a(p) \tag{22}$$

$$= -\int d^3p\; a(p)\,\delta^3(\vec{p}-\vec{k}) = -a(k) \tag{23}$$

To show that this counts the number of particles in a state, consider the following state,

$$|p_1, p_2, \ldots, p_n\rangle = a^\dagger(p_n)\cdots a^\dagger(p_2)a^\dagger(p_1)\,|0\rangle \tag{24}$$

Clearly,

$$\begin{aligned}
\mathcal{N}\,|p_1, p_2, \ldots, p_n\rangle &= \mathcal{N} a^\dagger(p_n) a^\dagger(p_{n-1}) \cdots a^\dagger(p_2) a^\dagger(p_1)\,|0\rangle && (25)\\
&= [\mathcal{N}, a^\dagger(p_n)] a^\dagger(p_{n-1}) \cdots a^\dagger(p_2) a^\dagger(p_1)\,|0\rangle \\
&\qquad + a^\dagger(p_n) \mathcal{N} a^\dagger(p_{n-1}) \cdots a^\dagger(p_2) a^\dagger(p_1)\,|0\rangle && (26)\\
&= |p_1, p_2, \ldots, p_n\rangle + a^\dagger(p_n)\mathcal{N}\,|p_1, p_2, \ldots, p_{n-1}\rangle && (27)\\
&= 2\,|p_1, p_2, \ldots, p_n\rangle + a^\dagger(p_n) a^\dagger(p_{n-1})\mathcal{N}\,|p_1, p_2, \ldots, p_{n-2}\rangle && (28)\\
&\vdots \\
&= n\,|p_1, p_2, \ldots, p_n\rangle + a^\dagger(p_n) a^\dagger(p_{n-1}) \cdots a^\dagger(p_2) a^\dagger(p_1)\mathcal{N}\,|0\rangle && (29)\\
&= n\,|p_1, p_2, \ldots, p_n\rangle && (30)
\end{aligned}$$

where $\int d^3p\; a^\dagger(p) a(p)\,|0\rangle = 0$ has been used.

There is one term we get for commuting $\mathcal{N}$ past each creation operator, therefore we are counting the number of creation operators, and hence the number of particles in the state.