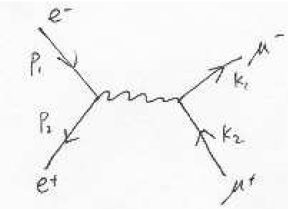
Homework 10 Solutions

10.1 - Helicity/chirality amplitudes for $e^+e^- \to \mu^+\mu^-$



(i) (a) $e_L^- + e^+ \to \mu_L^- + \mu^+$

$$M = \frac{e^2}{s}\left[\bar{v}(p_2)\gamma^\mu L\, u(p_1)\right]\left[\bar{u}(k_1)\gamma_\mu L\, v(k_2)\right]$$

$$\Rightarrow |M|^2 = \frac{e^4}{s^2}\sum_{\text{spins}}\left(\bar{v}(p_2)\gamma^\mu L\, u(p_1)\bar{u}(p_1)\gamma^\nu L\, v(p_2)\right)$$
$$\times \sum_{\text{spins}}\left(\bar{u}(k_1)\gamma_\mu L\, v(k_2)\bar{v}(k_2)\gamma_\nu L\, u(k_1)\right)$$

The electron part $= tr\left[\not{p}_2\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\not{p}_1\gamma^\nu\left(\frac{1-\gamma_5}{2}\right)\right]$ (we ignore masses)

$$= tr\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\left(\frac{1-\gamma_5}{2}\right)\right]$$

Use the trace formula : $tr\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

and $tr\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5\right] = -4i\varepsilon^{\mu\nu\rho\sigma}$

we get for the electron part $= 2(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu}p_2\cdot p_1 + i\varepsilon^{\alpha\mu\beta\nu}p_{2,\alpha}p_{1,\beta})$

Similarly, the muon part $= 2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - g^{\mu\nu}k_1\cdot k_2 + i\varepsilon^{\alpha\mu\beta\nu}k_{1,\alpha}k_{2,\beta})$

Dotting these two parts

$$\Rightarrow |M|^2 = \frac{4e^4}{s^2}\left[2(p_2\cdot k_1)(p_1\cdot k_2) + 2(p_1\cdot k_1)(p_2\cdot k_2) - \varepsilon^{\alpha\mu\beta\nu}\varepsilon_{\rho\mu\sigma\nu}p_{2,\alpha}p_{1,\beta}k_1^\rho k_2^\sigma\right]$$

recall that $\varepsilon^{\alpha\mu\beta\nu}\varepsilon_{\rho\mu\sigma\nu} = -2(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)$

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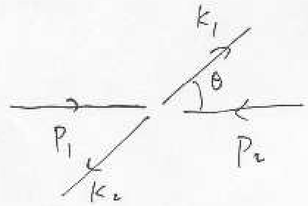
$$\Rightarrow |\mathcal{M}|^2 = \frac{8e^4}{s^2}\left[(P_2\cdot k_1)(P_1\cdot k_2) + \cancel{(P_1\cdot k_1)(P_2\cdot k_2)} + (P_2\cdot k_1)(P_1\cdot k_2) - \cancel{(P_2\cdot k_2)(P_1\cdot k_1)}\right]$$

$$= \frac{16e^4}{s^2}(P_2\cdot k_1)(P_1\cdot k_2)$$



From kinematics,

$$P_1\cdot k_2 = \frac{s}{4}(1+\cos\theta)$$

$$P_2\cdot k_1 = \frac{s}{4}(1+\cos\theta)$$

$$\Rightarrow |\mathcal{M}|^2 = e^4(1+\cos\theta)^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(e_L^- + e^+ \to \mu_L^- + \mu^+) = \frac{\alpha^2}{4E_{cm}^2}(1+\cos\theta)^2$$

(b) $e_L^- + e^+ \to \mu_R^- + \mu^+$

The only difference with respect to the previous part is the sign of γ_5 term for the muon part

$$\Rightarrow |\mathcal{M}|^2 = \frac{4e^4}{s^2}\left[2(P_2\cdot k_1)(P_1\cdot k_2) + 2(P_1\cdot k_1)(P_2\cdot k_2) + \varepsilon^{\alpha\mu\beta\nu}\varepsilon_{\rho\mu\sigma\nu}P_{2,\alpha}P_{1,\beta}k_1^\rho k_2^\sigma\right]$$

$$= \frac{16e^4}{s^2}(P_1\cdot k_1)(P_2\cdot k_2)$$

From kinematics, $P_1\cdot k_2 = P_2\cdot k_2 = \frac{s}{4}(1-\cos\theta)$

$$\Rightarrow |\mathcal{M}|^2 = e^4(1-\cos\theta)^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(e_L^- + e^+ \to \mu_R^- + \mu^+) = \frac{\alpha^2}{4E_{cm}^2}(1-\cos\theta)^2$$

(3)

(c) $e_R^- + e^+ \to \mu_L^- + \mu^+$

Comparing to part (a), the only difference is the sign of γ_5 for electron part. Therefore the result is the same as part (b)

$$\Rightarrow \frac{d\sigma}{d\Omega}(e_R^+ + e^+ \to \mu_L^- + \mu^+) = \frac{\alpha^2}{4E_{CM}^2}(1-\cos\theta)^2$$

(d) $e_R^- + e^+ \to \mu_R^- + \mu^+$

Comparing to part (a), the sign of γ_5 term in both electron & muon parts are changed, so the result is the same as part (a)

$$\Rightarrow \frac{d\sigma}{d\Omega}(e_R^- + e^+ \to \mu_R^- + \mu^+) = \frac{\alpha^2}{4E_{CM}^2}(1+\cos\theta)^2$$

(ii) (a) For the electron part: $\bar{v}(p_2)\gamma^\mu L\, u(p_1) = v^\dagger(p_2) L\, \gamma^0\gamma^\mu u(p_1)$

$$= \overline{(L\,v(p_2))}\,\gamma^\mu u(p_1)$$

LH v spinor $\Rightarrow$ RH positron

similarly, the antimuon is also RH.

The process is: $e_L^- + e_R^+ \to \mu_L^- + \mu_R^+$.

Carry out the same analysis for the other three processes:

(b) $e_L^- + e_R^+ \to \mu_R^- + \mu_L^+$

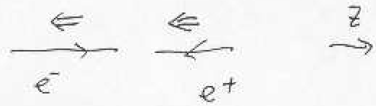
(c) $e_R^- + e_L^+ \to \mu_L^- + \mu_R^+$

(d) $e_R^- + e_L^+ \to \mu_R^- + \mu_L^+$

(iii) For process (a): $e_L^- + e_R^+ \to \mu_L^- + \mu_R^+$

The differential cross section vanish when $\cos\theta = -1$, i.e., along the backward direction. Let's analyze this process using angular momentum conservation.

Initial state



total angular momentum along z direction $\Leftarrow$, note there is no orbital angular momentum along z direction

Final state

total angular momentum along z direction $\Rightarrow$

We can see that the angular momentum along z direction cannot be conserved, therefore final state muon cannot go backwards with respect to initial electron. This confirms our result by calculating differential cross section.

(iv) The sum gives us

$$\frac{d\sigma_{total}}{d\Omega}(e^- + e^+ \to \mu^- + \mu^+) = \frac{\alpha^2}{E_{CM}^2}\left(1+\cos^2\theta\right)$$

Average over the initial state polarization

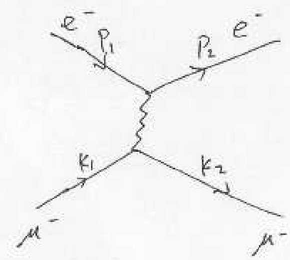
$$\Rightarrow \frac{d\sigma_{unpolarized}}{d\Omega}(e^- + e^+ \to \mu^+ + \mu^-) = \frac{\alpha^2}{4E_{CM}^2}\left(1+\cos^2\theta\right)$$

This agrees with the result of unpolarized differential cross-section.

This is expected since for these 4 processes, their initial or final states are different, therefore there cannot be interference among them.

10.2- Electron-muon elastic scattering

$e^- + \mu^- \to e^- + \mu^-$



$$\mathcal{M} = \frac{e^2}{t}\, \bar{u}(p_2)\gamma^\mu u(p_1)\, \bar{u}(k_2)\gamma_\mu u(k_1)$$

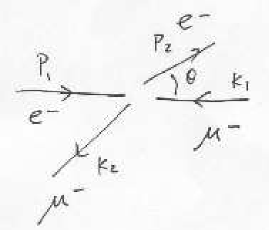
where $t = (p_1 - p_2)^2$

$$\frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{4t^2}\, tr[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu]\, tr[\not{k}_2 \gamma_\mu \not{k}_1 \gamma_\nu]$$

where we ignored masses, Use the trace formula

$$\Rightarrow \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{4e^4}{t^2}\left(p_2^\mu p_1^\nu + p_2^\nu p_1^\mu - g^{\mu\nu} p_1\cdot p_2\right)\left(k_{2,\mu}k_{1,\nu} + k_{2,\nu}k_{1,\mu} - g_{\mu\nu}\, k_1\cdot k_2\right)$$

$$= \frac{8e^4}{t^2}\left[(p_1\cdot k_1)(p_2\cdot k_2) + (p_1\cdot k_2)(p_2\cdot k_1)\right]$$



From kinematics, we have

$$p_1\cdot k_1 = p_2\cdot k_2 = \frac{s}{2}$$

$$p_1\cdot k_2 = p_2\cdot k_1 = (1+\cos\theta)\frac{s}{4}$$

$$t = (p_1 - p_2)^2 = -2p_1\cdot p_2 = -\frac{s}{2}(1-\cos\theta)$$

$$\Rightarrow \frac{1}{4}\sum_{spin}|\mathcal{M}|^2 = \frac{32e^4}{s^2(1-\cos\theta)^2}\left[\frac{s^2}{4} + (1+\cos\theta)^2\frac{s^2}{16}\right]$$

$$= \frac{2e^4}{(1-\cos\theta)^2}\left[4 + (1+\cos\theta)^2\right]$$

$$\Rightarrow \frac{d\sigma\,(e^-+\mu^- \to e^-+\mu^-)}{d\Omega} = \frac{\alpha^2}{2E_{cm}^2(1-\cos\theta)^2}\left[4 + (1+\cos\theta)^2\right]$$

Total cross section $\sigma_{tot} = \int\left(\frac{d\sigma}{d\Omega}\right)\cdot d\Omega$, which diverges. This is due to long range Coulomb interacti

(6)

Comparing to $e^+e^- \to \mu^+\mu^-$:

$$\frac{1}{4}\sum_{spin} \left|\mathcal{M}(e^+e^- \to \mu^+\mu^-)\right|^2 = \frac{8e^4}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right]$$

$$\frac{1}{4}\sum_{spin} \left|\mathcal{M}(e^-\mu^- \to e^-\mu^-)\right|^2 = \frac{8e^4}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right]$$

where we expressed the unpolarized amplitude squared in terms of Mandelstam variables. Under crossing symmetry, $s \leftrightarrow t$, we can see that these two processes are indeed related to each other by crossing symmetry.

10.3 - A number for cross-section

10.3.1 - Basic/benchmark cross-section

In problem 10.1, we calculated the total unpolarized differential cross-section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{CM}^2}\left(1+\cos^2\theta\right) \tag{1}$$

Therefore, the total cross-section is,

$$\begin{aligned}
\sigma &= \int d\Omega\, \frac{d\sigma}{d\Omega} &(2)\\
&= \int_0^{2\pi} d\phi \int_0^1 d(\cos\theta)\, \frac{\alpha^2}{4E_{CM}^2}\left(1+\cos^2\theta\right) &(3)\\
&= \frac{2\pi\alpha^2}{3E_{CM}^2} &(4)
\end{aligned}$$

At the center-of-mass energy of 100 GeV, the cross-section is,

$$\sigma = \frac{2\pi}{3(100)^2}\left(\frac{1}{135}\right)^2 \text{ GeV}^{-2} = 1.149\times 10^{-8} \text{ GeV}^{-2} \tag{5}$$

This is in natural units, where area is of dimensions $[M]^{-2}$. To convert it into m^2, say, we can supply factors of $\hbar$ and c.

$$\begin{aligned}
\sigma &= 1.149\times 10^{-8}(\hbar c)^2 \text{ GeV}^{-2} &(6)\\
&= 1.149\times 10^{-8}\left[\frac{\hbar c}{\text{GeV}}\right]^2\left[\frac{1 \text{ GeV}}{10^9\times 1.6\times 10^{-19} \text{ kg m}^2/\text{s}^2}\right]^2\left[\frac{1.05\times 10^{-34} \text{ kg m}^2/\text{s}}{\hbar}\right]^2\left[\frac{3\times 10^8 \text{ m/s}}{c}\right]^2 &(7)\\
&= 4.45\times 10^{-40} \text{ m}^2 &(8)
\end{aligned}$$

The more commonly employed unit for cross-sections in particle physics is the "picobarn", where 1 pb = 10^{-40} m^2. Then,

$$\sigma = 4.45 \text{ pb} \tag{9}$$

10.3.2 - Scaling of cross-section with charge and energy

Part (i)

The cross-section scales as $1/E_{CM}^2$ (which you could have guessed without any calculation! When electrons and muons are massless, then there is only one scale E_{CM} in the problem to provide the dimensions of cross-section). So the cross-section will be twice at center of mass energy $E_{CM} = \frac{1}{\sqrt{2}}100$ GeV $= 70.71$ GeV.

Part (ii)

The cross-sections scales as α^2, or in other words, the amplitude scales as $\alpha(\sim e^2)$. The two powers of coupling constant (or charge) arise from the charge of the electron and the charge of the muon, which are both equal to e.

Therefore, for down-quark (coupling $-\frac{e}{3}$) annihilating to up-quark (coupling $\frac{2e}{3}$), the amplitude will be $\frac{2}{9}$ of the amplitude for $e^+e^- \to \mu^+\mu^-$.

$$\sigma_{d\bar{d}\to u\bar{u}} = \left(\frac{2}{9}\right)^2 \sigma_{e^+e^-\to\mu^+\mu^-} \tag{10}$$

at the same energy.

Thus, the quark annihilation cross-section at $\frac{2}{9} \times 100$ GeV $= 22.2$ GeV will be the same as the electron-muon cross-section at 100 GeV.

Part (iii)

If the fundamental charge was twice as large, α would be four times larger. Consequently, the cross-sections in part (i) and part(ii) would be 16 times larger. Numerically, the cross-section in part (i) would become,

$$\sigma = 16 * 4.45 \text{ pb} = 71.2 \text{ pb} \tag{11}$$