Spin of single (anti) particle states

- Of course spin of particle as wavefunction is $\frac{1}{2}$ – that's the way we derived DE!
- But, we have come a bit far from it: first Lagrangian for classical field such that EOM is DE; then quantize to get particle states, i.e., $f^{\dagger} \ldots \hat{f}^{\dagger} |0\rangle \ldots$
- spin of above 1 (anti-) particle state: of course, for given $\bar{k}$, there are 2 particle states ($f^{\dagger}_{s=1,2}(\bar{k})|0\rangle$) and similarly 2 anti-particle states ($\hat{f}^{\dagger}_{s=1,2}(\bar{k})|0\rangle$ so that we expect these particles & anti-particles to have spin $\frac{1}{2}$ (i.e., the 2 states mentioned above are spin $= \pm \frac{1}{2}$ along some axis), but now we wish to show it explicitly...
- We'll follow approach similar to that for spatial/linear momentum of the states $a^{\dagger}_{p} \ldots a^{\dagger}_{p'} |0\rangle$ of scalar field, i.e., we constructed conserved (Noether) charge associated with spatial translations: $P_i = -\int d^3x \, \Pi \, \partial_i \phi_i \ldots$ which becomes an operator upon quantizing ϕ with $\bar{P}(a^{\dagger}_{p} \ldots a^{\dagger}_{p'} |0\rangle) = (\bar{p} + \bar{p}' + \ldots)(a^{\dagger}_{p} \ldots a^{\dagger}_{p'} |0\rangle)$

- So, we first determine the conserved charge associated with spatial rotations ... which becomes angular momentum operator (denoted by $\boxed{Q_J^{i=1,2,3}}$) upon quantizing ψ. Then, we determine its eigenvalues when action (i.e., acting) on $f^{\dagger}_{s=1,2}(\bar{k})|0\rangle$ and $\hat{f}^{\dagger}_{s=1,2}(\bar{k})|0\rangle$

(Note: PS & LP use J for Q_J, whereas in lecture J was used for spinor rotation ...

- It suffices to consider to consider projection of $Q_J^{i=1,2,3}$ along $\bar{k}$ (direction of motion of particle/anti-particle) since any orbital angular momentum will then not contribute, i.e., we'll get spin (along direction of motion, i.e., helicity)

- Without loss of generality, choose $\boxed{\bar{k} = (0,0,k^3)}$ with $k^3 > 0$ so that we focus on $\boxed{Q_J^{z \text{ or } 3}}$, i.e., rotation about z-axis : set $\omega_{12} = -\omega_{21} = \theta$ (with rest of ω's - the Lorentz transformation parameters to be zero) so that

e.g., $\frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$ in Weyl representation

$$\psi'(x) \approx \underbrace{(1 - i\theta J^3)}_{\text{infinitesimal form of above } \Lambda_{1/2}} \psi(\Lambda_{1/2}^{-1} x)$$

with $(\Lambda_{1/2}^{-1} x)$ being $(x + \theta y, y - \theta x, z)$

- However, to apply Noether's theorem, we need shift in field at a fixed point, i.e.,

$$\left(\Delta\psi(x) =\right) \overbrace{\psi'(x) - \psi(x)}^{\text{same}} \text{ which is } \simeq -\theta\left(x\partial_y - y\partial_x + iJ^3\right)\psi(x)$$

Thus, time component of conserved current is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\left(\frac{\Delta\psi}{\theta}\right) = -i\bar{\psi}\gamma^0\left(x\partial_y - y\partial_x + iJ^3\right)\psi$$

i.e., $\boxed{Q_J^z} = \int d^3x\, \psi^\dagger\left[-i(x\partial_y - y\partial_x) + J^3\right]\psi$

$\boxed{\text{Claim}}$: choose the 1 particle & 1 antiparticle states to be the ones associated with spinors $u_\pm(k^3)$ and $v_\pm(k^3)$ discussed earlier. Then,

$$Q_J^z\left(f_\pm^\dagger(k^3)|0\rangle\right) = \left(\pm\frac{1}{2}\right)\left(f_\pm^\dagger(k^3)|0\rangle\right) \qquad \text{case 1}$$

$$Q_J^z\left(\hat{f}_\pm^\dagger(k^3)|0\rangle\right) = \left(\pm\frac{1}{2}\right)\left(\hat{f}_\pm^\dagger(k^3)|0\rangle\right) \qquad \text{case 2}$$

Note that the helicity of 1 antiparticle state, i.e., $\left(\hat{f}_\pm^\dagger(k^3)|0\rangle\right)$ matches the label "$\pm$" on spinor v, whereas <u>helicity</u> of the <u>spinor</u> is actually opposite (for $k^3>0$), i.e., $J^3 v_\pm(k^3) = \mp\frac{1}{2} v_\pm(k^3)$ or (equivalently) v_+ & v_- are constructed using 2-component spinors $\begin{pmatrix}0\\1\end{pmatrix}$ & $\begin{pmatrix}1\\0\end{pmatrix}$ which have σ^3 eigenvalues -1 and $+1$ (respectively)

$\boxed{\text{Proof}}$ Since vacuum has no angular momentum, i.e., $Q_J^z |0\rangle = 0$, we see that

$$Q_J^z \left(f_r^+(k^3)|0\rangle \right) = \left[Q_J^z, f_r^+(k^3) \right]_{-} |0\rangle$$

$\leftarrow$ commutator

(and similarly for $\hat{f}^+ \ldots$) so that it suffices to calculate action of commutator on vacuum

"Outline" of strategy

- Express Q_J^z in terms of f^+, $\hat{f}^+$ etc. by writing $\psi^\dagger, \psi$ in terms of these operators
- For the case of commutator of Q_J^z with $f^+(k^3)$, take the 2^{nd} term, i.e., $\ominus\, f^+(k^3)\, Q_J^z$, and "move" $f_r^+(k^3)$ to right (using anti-commutators of f, $\hat{f}$ etc.) so that it cancels 1^{st} term ... expect for a non-vanishing contribution from anti-commutator of $f_r^+(k^3)$ with a $f_s(\vec{p})$ contained in ψ part of Q_J^z

  [for commutator of Q_J^z with $\hat{f}^+(k^3)$, it's more convenient to take 1^{st} term, i.e., $\oplus\, Q_J^z \hat{f}^+(k^3)$ and move $\hat{f}_r^+(k^3)$ to left ...]
- For both $f_r^+(k^3)$ & $\hat{f}_r^+(k^3)$ cases, the "orbital" part of Q_J^z, i.e., $x\partial_y - y\partial_x$, doesn't contribute (as expected) so that we are left with J^3 part

— Between the cases $f_r^+(k^3)$ & $\hat{f}_r^+(k^3)$, there are 2 sign flips as follows (i) in manipulating the commutator as above — note that only non-vanishing contribution in case of $\hat{f}_r^+(k^3)$ comes from anti-commutator of $\hat{f}_r^+(k^3)$ with ψ^+ in Q_J^z [cf. case of $f_r^+(k^3)$ discussed above]

(ii) $J_3\, u_\pm(k^3) = \pm\frac{1}{2} u_\pm(k^3)$ vs. $J_3\, v_\pm(k^3) = \mp\frac{1}{2} v_\pm(k^3)$

$\Rightarrow$ no net sign flip between the 2 cases (as seen in the claim above)

"Gory" details of proof

We have $Q_J^z = \int d^3x\, \psi^+\left[-i(x\partial_y - y\partial_x) + J^3\right]\psi$

with $\psi(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \sum_{s=1,2}\left[f_s(\bar{p})\, u_s(\bar{p})\, e^{-ip.x} + \hat{f}_s^+(\bar{p})\, v_s(\bar{p})\, e^{+ip.x}\right]$

and $\psi^+(x) = \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} \sum_{s'=1,2}\left[f_{s'}^+(\bar{p}')\, u_{s'}^+(\bar{p}')\, e^{+ip'.x} + \hat{f}_{s'}(\bar{p}')\, v_{s'}^+(\bar{p}')\, e^{-ip'.x}\right]$

Case 1, i.e., evaluate $[Q_J^z, f_r^+(k^3)]_-|0\rangle$

Work on its 2^{nd} term, i.e., $-\, f_r^+(k^3)\, Q_J^z|0\rangle$. Move f_r^+ to the right, picking up (-1) when going past ψ^+ and another (-1) when going past ψ, i.e., net $+1$ so as to cancel 1^{st} term of $[\cdots]_-|0\rangle$

...except for the contribution from

$$\left[f_s(\bar{p}),\ f_r^{+}(\bar{k}) \right]_{\oplus} = \delta_{rs}\, \delta^3(\bar{p}-\bar{k})$$

from ψ in Q_J^z

of course only $k^3 \neq 0$ here

[again, other anticommutators of $f_p^{+}(k^3)$, i.e., with $f_{s'}^{+}$, $\hat{f}_{s'}$ in ψ^{+} and $\hat{f}_s^{+}$ in ψ are zero]

Use above $\delta^3(\bar{p}-\bar{k})$ to do $\int d^3p$ so that

No "−" here since taking "−" from 2nd term in $[\cdots]_-|0\rangle$ "cancels" that from moving f^{+}...

$$\left[Q_J^z,\ f_r^{+}(\bar{k}) \right]_- |0\rangle = \int \frac{d^3x}{(2\pi)^3} \int \frac{d^3p'}{2\sqrt{E_{p'}E_k}}$$

due to $\bar{p}=\bar{k}$

$\psi^{\oplus}$ part

$$\times \sum_{s'} \Big[f_{s'}^{+}(\bar{p}')\, u^{+}(\bar{p}')\, e^{+ip'\cdot x} + \underbrace{\hat{f}_{s'}(\bar{p}')\, v_{s'}^{+}(\bar{p}')\, e^{-ip'\cdot x}\Big] |0\rangle}_{\text{gives } 0}$$

$$\times \left[-i\left(x\, \partial_y - y\, \partial_x\right) + J^3 \right] u_r(\bar{k})\, e^{-ik\cdot x}$$

(i) "J^3 part": use $\delta^3(\bar{p}'-\bar{k})$ from $\displaystyle\int \frac{d^3x}{(2\pi)^3}\, e^{+ip'\cdot x}\, e^{-ik\cdot x}$

to do $\int d^3p'$, giving

$\bar{p}'=\bar{k}$ sets $E_{p'} = E_k$ so that $e^{iE..t}$'s cancel

$$\left[Q_J^z,\ f_r^{+}(\bar{k}) \right]_- |0\rangle = \frac{1}{2E_k} \sum_{s'=1,2} \overbrace{f_{s'}^{+}(\bar{k})|0\rangle}\, u_s^{+}(k)\, J_3\, u_r(\bar{k})$$

Choose $r = \pm$ for $\bar{k}=(0,0,k^3)$ so that $J_3 u_r = \pm\frac{1}{2} u_r$ and the normalization condition $u_r^{+}(\bar{p})\, u_{s'}(\bar{p}) = 2E_p\, \delta_{rs'}$ sets $s' = \pm$ (respectively), i.e., $\left[Q_J^z, f_{\pm}^{+}(k^3) \right]_- |0\rangle = \left(\pm\frac{1}{2}\right)\left(f_{\pm}^{+}(k^3)|0\rangle\right)$

as desired since

(ii) $(x\partial_y - y\partial_x)$ part gives 0 due to it acting on $e^{+ik^3 \cdot z}$

→ again $\bar{k} = (0,0,k^3)$

Case 2, i.e., evaluate $[Q_J^z, \hat{f}_r^+(k^3)]_- |0\rangle$

Work on its 1st term, i.e., $\oplus\, Q_J^z \hat{f}_r^+(k^3)|0\rangle$.

Move $\hat{f}_r^+$ to the left ... giving a non-vanishing contribution to $[\cdots]_-|0\rangle$ only from $[\hat{f}_{s'}(\bar{p}'), \hat{f}_r^+(\bar{k})] = \delta_{rs'}\,\delta^3(\bar{p}'-\bar{k})$ ... so

← from $\psi^{\oplus}$ in Q_J^z

that

from moving $\hat{f}^+$ to left of ψ (cf. case 1 above)

$$[Q_J^z, \hat{f}_r^+(k)]_-|0\rangle = (-1)\int \frac{d^3x}{(2\pi)^3} \times \frac{d^3p}{2\sqrt{E_p E_k}} \times v_r^+(\bar{k})\,e^{-ik\cdot x}$$

→ due to $\bar{p}' = \bar{k}$

$$\left[-i(x\partial_y - y\partial_x) + J^3\right] \underbrace{\sum_{s=1,2}\left[f_s(\bar{p})\,u_s(\bar{p})\,e^{-ip\cdot x} + \hat{f}_s^+(\bar{p})\,v_s(\bar{p})\,e^{+ip\cdot x}\right]}_{\psi \text{ part}} |0\rangle$$

gives 0

(i) J^3 part: use $\delta^3(\bar{p}-\bar{k})$ from $\int \frac{d^3x}{(2\pi)^3}\, e^{-ik\cdot x + ip\cdot x}$ ....

so that $[Q_J^z, \hat{f}_r^+(\bar{k})]_-|0\rangle$ (J^3 part only)

$$= \frac{(-1)}{2E_k} \sum_{s=1,2} \left(v_r^+(\bar{k})\, J_3\, v_s(\bar{k})\right)\left(\hat{f}_s^+(\bar{k})\,|0\rangle\right)$$

Choose the basis $s = 1,2$ to be $s = +,-$ for $\bar{k} = (0,0,k^3)$

so that $J_3\, v_\pm(k^3) = \left(\mp \tfrac{1}{2}\right) v_\pm(k^3)$ (cf. for u above) and normalization condition for v's gives

$$[Q_J^z, \hat{f}_\pm^+(k^3)]_- |0\rangle \quad (J^3 \text{ part only})$$

$$= (-1)\left(\mp \tfrac{1}{2}\right)\left(\hat{f}_\pm^+(k^3)|0\rangle\right)$$

(ii) $(x\partial_y - y\partial_x)$ part : $\int dy\, e^{(+ik_y y - i p_y y)} \times (p_y)$

sets $p_y = k_y = 0$ [again, $\vec{k} = (0,0,k^3)$].

from ∂_y on $e^{+ip.x}$

Similarly, $\int dx\, e^{ik_x x - i p_x x} \times (p_x)$ sets $p_x = 0$.

Thus this part doesn't contribute.

So, we get (as desired)

$$[Q_J^z, \hat{f}_\pm^+(k^3)]_- |0\rangle = \left(\pm \tfrac{1}{2}\right)\left(\hat{f}_\pm^+(k^3)|0\rangle\right)$$