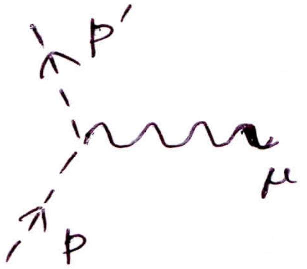
Illustrating $k^\mu M_\mu = 0$ using "scalar" Compton scattering

Scalar QED: $\mathcal{L} = (D^\mu \phi)^\dagger (D_\mu \phi) - m^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

where $D_\mu = \partial_\mu + ie A_\mu$, i.e., ϕ destroys spin-0 particle of charge -1 (or creates spin-0 particle of charge $+1$)...

Feynman rules for $\boxed{\text{vertices}}$ (propagators same as free case): canonical quantization is a bit tricky due to derivative interaction, i.e., $\partial_\mu \phi^\dagger ie A_\mu \phi$ [$\Rightarrow$ (at least (naively)), expression for conjugate momentum of ϕ, i.e., $\partial \mathcal{L} / \partial (\underline{\underline{\partial_0}} \phi)$ is modified] : easier to do it using functional integral formalism (see problem 9.1 in Peskin & Schroeder) with the result:



$= -ie(p+p')^\mu$


$= 2ie^2 g^{\mu\nu}$

[Note (i) arrow on scalar line in 1st vertex denotes flow of negative charge and (ii) factor of 2 in 2nd vertex is a combinatorial one due to 2 identical particles/fields, i.e., photons, at the same vertex]

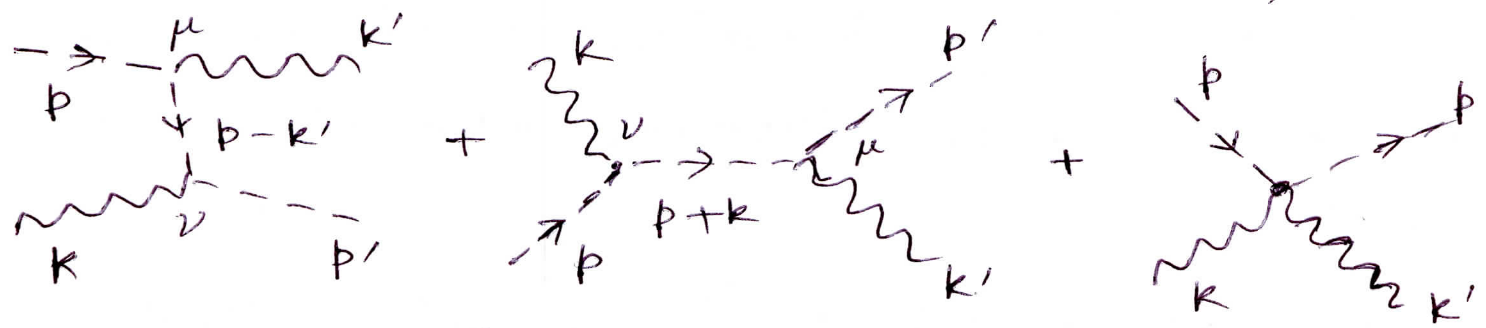
"Compton" scattering in scalar QED :

$$\phi^-(p) + \gamma(k, r) \longrightarrow \phi^-(p') + \gamma(k', r')$$

(spin-0 particle of charge -1)

denotes polarization

$\underline{\underline{3}}$ diagrams (2 similar to $e^- + \gamma \to e^- + \gamma$, i.e., due to 1st vertex above ; 3rd due to 2nd vertex above is "new" – we'll see that the 3rd diagram is required for consistency, as expected from gauge invariance of interaction terms : without $A_\mu^2 \phi^\dagger \phi$, theory is not gauge-invariant) :



$$\Rightarrow i\mathcal{M} = (ie)^2\, i \left\{ \frac{\left[(p+p-k')^\mu \epsilon^*_{r'\mu}(k')\right]\left[(p+p'-k')^\nu \epsilon_{r\nu}(k)\right]}{\left[(p-k')^2 - m^2\right]} \right.$$

1st diagram

m ↳ scalar mass

$$\left. + \frac{\left[(p+p'+k)^\mu \epsilon^*_{r'\mu}(k')\right]\left[(2p+k)^\nu \epsilon_{r\nu}(k)\right]}{(p+k)^2 - m^2} \right\}$$

2nd diagram

$$+\, 2ie^2\, \epsilon_{r\nu}(k)\, \epsilon^{*\nu}_{r'}(k')$$ ← 3rd diagram

We would like to show $k_\nu M^\nu = 0$, where M^ν is above Feynman amplitude, but without $\epsilon_{r\nu}(k)$. Use (i) $p^2 = p'^2 = m^2$ (on-shellness of external scalars), (ii) $k^2 = k'^2 = 0$ (on-shellness of external photons), (iii) $p + k = p' + k'$ (overall energy-momentum conservation) and (iv) $k_\mu \epsilon^\mu_r(k) = k'_\nu \epsilon^{*\nu}_{r'}(k') = 0$ (Lorenz gauge)

used (iv) here — used (iii) here — ϵ_ν replaced by k_ν

$$k_\nu M^\nu = -e^2 \Big\{ [2p \cdot \epsilon'^*] [(2p' - k) \cdot k] / [(p'-k)^2 - m^2]$$

use (ii) — used (iii) here — $\epsilon_\nu \to k_\nu$

$$+ [(2p + k) \cdot k] [(2p' + k') \cdot \epsilon'^*] / (2p \cdot k) \Big\}$$

use (ii) — used (iii) here — use (iv) — used (i), (ii) here

$$+ 2e^2 k \cdot \epsilon'^*$$

$\epsilon_\nu \to k_\nu$

[we have used shorthand : $\epsilon_\nu \equiv \epsilon_{r\nu}(k)$ and $\epsilon'_\mu = \epsilon_{r'\mu}(k')$]

used (i) here

$$= -e^2 \Big\{ (2p \cdot \epsilon'^*)(2p' \cdot k)/(-2p' \cdot k) + (2p \cdot k)(2p' \cdot \epsilon'^*)/(2p \cdot k) \Big\} + 2e^2 k \cdot \epsilon'^*$$

3rd diagram is crucial!

$$= 2e^2 (p - p' + k) \cdot \epsilon'^* = 2e^2 k' \cdot \epsilon'^* = \underline{\underline{0}}$$

use (iii) here — use (iv) here