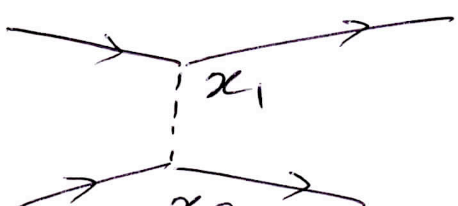
Fermion scattering: $e^-(p_1) + e^-(p_2) \to e^-(p_1') + e^-(p_2')$

- Why this example?
  - different than decay
  - new physics feature (will become part of Feynman rules): (−1) between diagrams due to identical fermions in initial/final state: expected from Fermi-Dirac statistics, but is "built-in" (like energy-momentum conservation)
  - as mentioned earlier, confirm that there are 4 possibilities in



i.e., (i) $e^-(p_1)$ destroyed at x_1, $e^-(p_1')$ created at x_1

[of course, then $e^-(p_2)$ has to be destroyed at x_2 and $e^-(p_2')$ created at x_2]

(ii) $e^-(p_1)$ destroyed and $e^-(p_1')$ created at x_2 ...

(iii) $e^-(p_1)$ destroyed and $e^-(p_2')$ created at x_1 ...

(iv) $e^-(p_1)$ destroyed and $e^-(p_2')$ created at x_2 ...

... with (i) & (ii) and similarly (iii) & (iv) giving same contribution after $\int d^4x_1 \int d^4x_2 \ldots$

- good news: last brute-force calculation since we'll develop and from now on use Feynman rules instead!

- no $S^{(1)}$, i.e., $\mathcal{H}_I^1$ as discussed earlier

- $S^{(2)} = \frac{(-ih)^2}{2!} \int d^4x_1 \int d^4x_2 \, T\left[:(\bar{\psi}^\alpha \psi^\alpha \phi)_{x_1}: (\bar{\psi}^\beta \psi^\beta \phi)_{x_2} \right]$

$\boxed{\text{Step 1}}$

- As per general idea, only terms in Wick expansion with 2ϕ contracted contribute (again, we need 2ψ, $2\bar{\psi}$'s to "take care of "external states), i.e.,

$$\ominus \; \underbrace{\phi(x_1)\,\phi(x_2)}_{} \left(\bar{\psi}_-^\alpha(x_1)\, \bar{\psi}_-^\beta(x_2)\, \psi_+^\alpha(x_1)\, \psi_+^\beta(x_2) \right)$$

due to moving $\bar{\psi}(x_2)$ to left of $\psi(x_1)$
(i.e., "contract")

- $\boxed{\text{Step 2}}$: Use 2ψ, $2\bar{\psi}$'s on external states

(i) initial state

$$\psi_+^\alpha(x_1)\, \psi_+^\beta(x_2)\, |e^-(p_1)\, e^-(p_2)\rangle$$

dummy...

$$= \int \frac{d^3k}{\sqrt{2E_k V}} \int \frac{d^3k'}{\sqrt{2E_{k'} V}} \, u^\alpha(k)\, u^\beta(k')\, e^{-ik\cdot x_1} e^{-ik'\cdot x_2}$$

$$\times \underbrace{f(k)\, f(k')}_{\text{from } 2\psi \text{ in } \mathcal{H}_I^2} \underbrace{f^\dagger(p_2) f^\dagger(p_1)\, |0\rangle}_{\text{from } |i\rangle}$$

New feature : choice of which ψ destroys which e^-... reflected above in the form of 2 terms

crucial

$$f(k)\, f(k')\, f^\dagger(p_2) f^\dagger(p_1)|0\rangle = \left[\delta^3(k'-p_2)\, \delta^3(k-p_1) \ominus \delta^3(k-p_2)\, \delta^3(k'-p_1) \right] \times |0\rangle$$

Proof of last relation (ignore spin/helicity index on $u, f, f^\dagger$ for simplicity):

$$f(k)\, f(k')\, f^\dagger(p_2) f^\dagger(p_1)|0\rangle = f(k)\left[\ominus f^\dagger(p_2) f(k') + \delta^3(p_2 - k')\right] \times f^\dagger(p_1)|0\rangle$$

$$= -f(k) f^\dagger(p_2)\left[-f^\dagger(p_1) f(k') + \delta^3(p_1 - k')\right]|0\rangle$$

$$+ \delta^3(p_2 - k')\left[-f^\dagger(p_1) f(k) + \delta^3(p_1 - k)\right]|0\rangle$$

$$= -\left[-f^\dagger(p_2) f(k) + \delta^3(p_2 - k)\right]\delta^3(p_1 - k')|0\rangle +$$

$$\delta^3(p_2 - k')\,\delta^3(p_1 - k)\,|0\rangle$$

$$= \left[\delta^3(p_2 - k')\,\delta^3(p_1 - k) \ominus \delta^3(p_2 - k)\,\delta^3(p_1 - k')\right]|0\rangle$$

$e^-(p_2)$ at x_2 ... $e^-(p_2)$ at x_1 ...

i.e., negative sign between 2 choices (cf. for spin-0 particles in HW 7)

$\Rightarrow$ initial state "contractions" give

"$E > 0$" plane-wave for destroyed particles

$$\frac{1}{\sqrt{2E_{p_1}V}}\frac{1}{\sqrt{2E_{p_2}V}}\left[u^\alpha(p_1)\, u^\beta(p_2)\, e^{-i(p_1\cdot x_1 + p_2\cdot x_2)} \ominus u^\alpha(p_2)\, u^\beta(p_1)\, e^{-i(p_2\cdot x_1 + ip_1\cdot x_2)}\right]|0\rangle$$

$$= \left(I_1^{\alpha\beta} - I_2^{\alpha\beta}\right)\times|0\rangle$$

$e^-(p_1)$ destroyed at x_2 ...

Similarly, final state contractions give

$$\frac{1}{\sqrt{2E_{p_1'}V}}\frac{1}{\sqrt{2E_{p_2'}V}}\langle 0|\Big[\bar{u}^{\alpha}(p_1')\,\bar{u}^{\beta}(p_2')\,e^{+i(p_1'\cdot x_1+p_2'\cdot x_2)}$$

$$\ominus\ \bar{u}^{\alpha}(p_2')\,\bar{u}^{\beta}(p_1')\,e^{+i(p_2'\cdot x_1+p_1'\cdot x_2)}\Big]$$

$\equiv \langle 0| \times (F_1^{\alpha\beta} - F_2^{\alpha\beta})$

$e^-(p_2')$ created at x_1 ...

$\boxed{\text{Step } ③}$ Combine ...

scalar contraction gives $i\Delta_F(x_1-x_2)$

$$S^{(2)} = -\frac{(-ih)^2}{2!}\int d^4x_1\int d^4x_2\,\overbrace{\int\frac{d^4q}{(2\pi)^4}\,i\,\Delta_F(q)\,e^{iq(x_1-x_2)}}^{i\Delta_F(x_1-x_2)}$$

$$\Delta_F(q) = \frac{1}{q^2-M^2}$$

$$\times\,(I_1^{\alpha\beta}-I_2^{\alpha\beta})(F_1^{\alpha\beta}-F_2^{\alpha\beta})\underbrace{\langle 0|0\rangle}_{1}$$

"Naively"

$\Rightarrow$ 4 terms of form $\bar{u}..\bar{u}..u..u$ from $I_1^{\alpha\beta}F_1^{\alpha\beta}$...

(corresponding to ④ possibilities mentioned earlier)

- However, $x_1 \Leftrightarrow x_2$ and $\alpha \Leftrightarrow \beta$ (all dummy variables) results in $I_1^{\alpha\beta} \Leftrightarrow +I_2^{\alpha\beta}$ and $F_1^{\alpha\beta} \Leftrightarrow +F_2^{\alpha\beta}$

$\Rightarrow$ doing it (only) on $\ominus I_2^{..}(F_1^{..}-F_2^{..})$ in $S^{(2)}$ above gives factor of 2 (multiplying $I_1^{..}(F_1^{..}-F_2^{..})$) (which cancels $\frac{1}{2!}$ from Taylor series ... as expected)

[Note: Since $\Delta_F(-q) = \Delta_F(q)$, we can simply do $q \to \ominus q$ as we do $x_1 \Leftrightarrow x_2$ in order to keep scalar propagator part the same]

$$\Rightarrow S^{(2)} = -(-ih)^2 \int d^4x_1 \int d^4x_2 \int \frac{d^4q}{(2\pi)^4} \, i\Delta_F(q) \, e^{iq\cdot(x_1-x_2)}$$

$$\times \; I_1^{\alpha\beta} \left(F_1^{\alpha\beta} - F_2^{\alpha\beta} \right)$$

For "1st term", i.e., $I_1^{\cdot\cdot} F_1^{\cdot\cdot}$

– $\boxed{\text{Step 4}}$ we have

$$\int d^4x_1 \, e^{i(\overbrace{q\cdot x_1 - p_1 x_1}^{\leftarrow \text{ from } I_1^{\cdot\cdot}} + \underbrace{p_1'\cdot x_1}_{\text{from } F_1^{\cdot\cdot}})} = (2\pi)^4 \delta^4(q - p_1 + p_1')$$

and $\int d^4x_2 \, e^{i(-q\cdot x_2 + p_2'\cdot x_2 - p_2 x_2)} = (2\pi)^4 \delta^4(q - p_2' + p_2)$

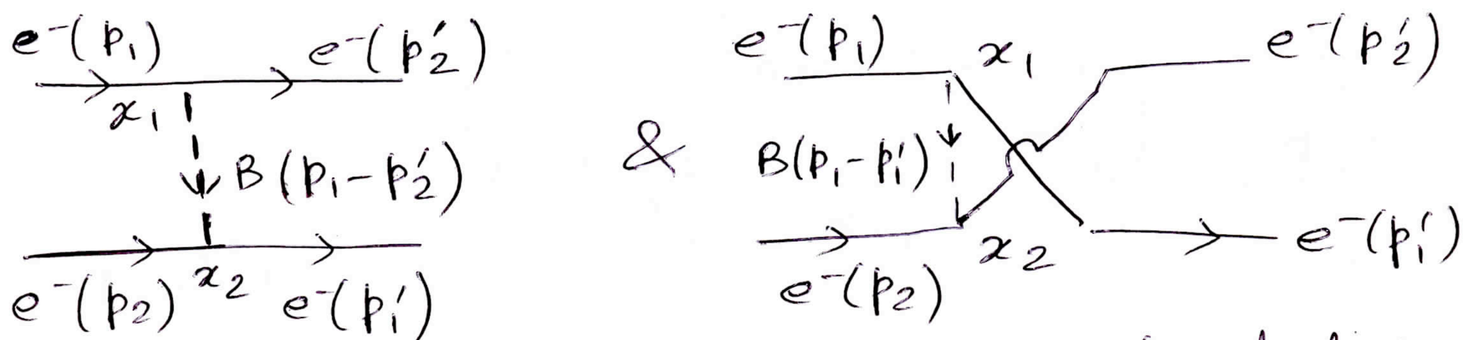
One combination of two δ^4's gives $\delta^4(p_1 - p_1' - p_2' + p_2)$, i.e., <u>overall</u> energy-momentum conservation (as expected); other combination "gets rid of" $\int d^4q$

... similarly for <u>2nd</u> term, i.e., $\ominus\, I_1^{\cdot\cdot} F_2^{\cdot\cdot}$ ... but with $p_1' \leftrightarrow p_2'$ ... to finally give

$$S^{(2)} = (-ih)^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2') \times \frac{1}{\sqrt{2E_{p_1}V}\sqrt{2E_{p_2}V}\sqrt{2E_{p_1'}V}\sqrt{2E_{p_2'}V}}$$

$$\times \Big[\overbrace{\bar{u}^\alpha(p_2')\,\bar{u}^\beta(p_1')\, i\Delta_F(p_1 - p_2')}^{\text{above 2nd term, i.e., } -I_1^{\cdot\cdot} F_2^{\cdot\cdot}} \ominus \underbrace{\bar{u}^\alpha(p_1')\,\bar{u}^\beta(p_2')\, i\Delta_F(p_1 - p_1')}_{\text{above 1st term, i.e., } I_1^{\cdot\cdot} F_1^{\cdot\cdot}} \Big] u^\alpha(p_1)\, u^\beta(p_2)$$

– Again, 1 term in Wick expansion gives 2 (different) terms in S-matrix element due to identical initial & final state particles ... two terms correspond to Feynman diagrams



(again, $x_1 \leftrightarrow x_2$ is not counted as separate diagram) ... where we show momentum flowing through scalar line [direction of arrow on it is really irrelevant since $\Delta_F(q) = \Delta_F(-q)$] : note that this momentum is different in 2 diagrams (as we might have expected – before calculation – based simply on 4-momentum conservation at each vertex) $\Rightarrow$ diagrams are really different!

– Finally, Fermi-Dirac statistics comes out automatically : 2nd term/diagram obtained from 1st one by $p_1' \leftrightarrow p_2'$ $\Rightarrow$ opposite sign (i.e., anti-symmetric under exchange of identical final state fermions)