2-to-2 scattering in center-of-mass (CM) frame

- Since calculation of σ is tedious, we'll simplify it by splitting into 2 parts (just like for decay rate):

(1) doing (part of) $\int dLIPS$ for general $\mathcal{M}_{fi}$

(2) plug-in / evaluate specific $\mathcal{M}_{fi}$ (i.e., for given $\mathcal{H}_I$)

Part (1), i.e.,

- $\int dLIPS$, in center-of-mass (CM) frame, i.e., total initial $\bar{p} = 0 \Rightarrow \bar{p}_1 = -\bar{p}_2$ (as in most high-energy physics experiments nowadays)

- step $\boxed{(i)}$: Use $\delta^3(\bar{p}_1 + \bar{p}_2 - \bar{p}_1' - \bar{p}_2')$ to do $\int d^3p_2'$

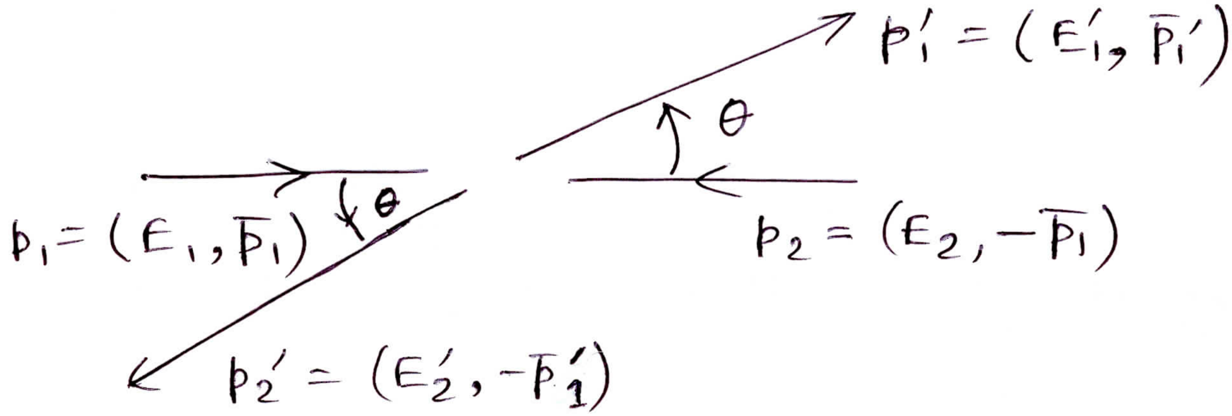
i.e., set $\bar{p}_2' = (\bar{p}_1 + \bar{p}_2) - \bar{p}_1' = -\bar{p}_1'$ in CM frame

then, we get

$$\sigma = \frac{1}{64\pi^2 E_1 E_2 v_{rel}} \int \frac{d^3p_1'}{E_1' E_2'} \delta(E_1 + E_2 - \underbrace{E_1'}_{\sqrt{|\bar{p}_1'|^2 + m_1'^2}} - \underbrace{E_2'}_{\sqrt{|\bar{p}_2'|^2 + m_2'^2} = \sqrt{|\bar{p}_1'|^2 + m_2'^2}})$$

$$\times |\mathcal{M}_{fi}|^2 \quad \leftarrow \text{set } \bar{p}_1 = -\bar{p}_2;\ \bar{p}_1' = -\bar{p}_2' \text{ here}$$

(Note that $E_1 \neq E_2$; $E_1' \neq E_2'$ in general even in CM frame)



– <u>step</u> $\boxed{\text{(ii)}}$ writing $\int d^3 p_1' = \int d|\bar{p}_1'|\, |\bar{p}_1'|^2 \underbrace{d\Omega}_{d\phi\, d(\cos\theta)}$, we see that above $\delta(E_1 + E_2 - E_1' - E_2')$ can be used to do $\int d|\bar{p}_1'|$, but <u>not</u> $\int d\Omega$ (just like for decay)

<u>sub-step</u> $\boxed{\text{(a)}}$ Solve for $|\bar{p}_1'|$:

We have $E_1 + E_2$ ($\equiv \sqrt{s}$, total initial energy: known)

$$= \sqrt{|\bar{p}_1|^2 + m_1^2} + \sqrt{\underbrace{|\bar{p}_2|^2}_{= |\bar{p}_1|^2} + m_2^2} \Rightarrow$$

$$\left(\sqrt{s} - \sqrt{|\bar{p}_1|^2 + m_1^2}\right)^2 = |\bar{p}_1|^2 + m_2^2$$

$$s + m_1^2 + \cancel{|\bar{p}_1|^2} - 2\sqrt{s}\sqrt{|\bar{p}_1|^2 + m_1^2} = \cancel{|\bar{p}_1|^2} + m_2^2$$

$$\Rightarrow |\bar{p}_1|^2 = \frac{(s + m_1^2 - m_2^2)^2}{4s} - m_1^2$$

Putting "primes" everywhere, except on $\sqrt{s}$ (i.e., using $E_1' + E_2' = E_1 + E_2 = \sqrt{s}$), gives

$$|\bar{p}_1'{}_{\text{(sol.)}}| = \sqrt{\frac{(s + m_1'^2 - m_2'^2)^2}{4s} - m_1'^2} \,,$$

where subscript "sol." on $\bar{p}_1'$ denotes
it being solution to the condition $E_1 + E_2 - E_1' - E_2' = 0$
(to be distinguished from variable $\bar{p}_1'$)

sub-step $\boxed{(b)}$ Use $\delta[f(x)] = \delta(x-a) \Big/ \left|\frac{df}{dx}\right|_{x=a}$

where $f(a) = 0$ to rewrite

$$\delta(E_1 + E_2 - E_1' - E_2') = \frac{\delta(|\bar{p}_1'| - |\bar{p}_1'{}_{sol.}|)}{\left|\partial E_1'/\partial|\bar{p}_1'| + \partial E_2/\partial|\bar{p}_1'|\right|_{\text{at } |\bar{p}_1'{}_{sol.}|}}$$

$$= \frac{\delta(|\bar{p}_1'| - |\bar{p}_1'{}_{sol.}|)}{\left. \frac{2|\bar{p}_1'|}{2\sqrt{|\bar{p}_1'|^2 + m_1'^2}} + \frac{2|\bar{p}_1'|}{2\sqrt{|\bar{p}_1'|^2 + m_2'^2}} \right| \text{at} \ldots}$$

$$= \delta(|\bar{p}_1'| - |\bar{p}_1'{}_{sol.}|) \times \frac{E_1' E_2'}{|\bar{p}_1'{}_{sol.}|(E_1' + E_2')}$$

Also, $v_{rel} = |\bar{p}_1| \left(\frac{1}{E_1} + \frac{1}{E_2}\right) = |\bar{p}_1| \frac{\sqrt{s}}{E_1 E_2}$

so that differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\bar{p}_1'{}_{sol.}|}{|\bar{p}_1|} |\mathcal{M}_{fi}|^2$$

use also $|\bar{p}_1'| = |\bar{p}_1'{}_{sol.}|$ here

- (cylindrical) symmetry (rotation about beam axis) $\Rightarrow |\mathcal{M}_{fi}|$ can't depend on ϕ $\Rightarrow \int d\phi = 2\pi$
- elastic limit (i.e., final = initial particles): set $m_1' = m_1$; $m_2' = m_2$ $\Rightarrow |\bar{p}_1'{}_{sol.}| = |\bar{p}_1| \Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |\mathcal{M}_{fi}|^2$

Inelastic scattering with 4-fermion interaction

— Consider the process

$$\nu_\mu(k) + e^-(p) \longrightarrow \mu^-(p') + \nu_e(k')$$

due to

$$\mathcal{L}_I = 2\sqrt{2}\, G_F \left(\underbrace{\overline{\Psi}_{\nu_e}}_{\text{creates } \nu_e} \gamma^\lambda L \Psi_e\right)\left(\overline{\Psi}_\mu \gamma_\lambda R \underbrace{\Psi_{\nu_\mu}}_{\text{destroys } \nu_\mu}\right)$$

where $L, R = \frac{1}{2}(1 \mp \gamma_5)$ are the usual chirality projection operators

(Note that Lahiri & Pal studies the case "L...L" instead of "L...R" above : see HW 8.3 for more discussion)

— By brute force (or Feynman rules with vertex factor being $\gamma^\lambda L$ etc.), we get

$$\boxed{\mathcal{M}_{fi}} = 2\sqrt{2}\, G_F \left[\bar{u}_{\nu_e}(k')\gamma^\lambda L\, u_e(p)\right] \times \left[\bar{u}_\mu(p')\gamma_\lambda R\, u_{\nu_\mu}(k)\right]$$

(no net Dirac/spinor index on each [ ])

— Note that $\bar{u}_{\nu_e} \gamma^\lambda L\, u_e = (u^\dagger_{\nu_e}\gamma^0) R \gamma^\lambda u_e$

(use $\gamma^\lambda \gamma_5 = -\gamma_5 \gamma^\lambda$)

$$= u^\dagger_{\nu_e} L \gamma^0 \gamma^\lambda u_e = (L u_{\nu_e})^\dagger \gamma^0 \gamma^\lambda u_e$$
$$= \overline{(L u_{\nu_e})}\, \gamma^\lambda u_e$$

i.e., ν_e created has L chirality; similarly created μ has R chirality

- Before embarking on calculation, let's see if there's a <u>physics expectation</u> (like we had for decay):

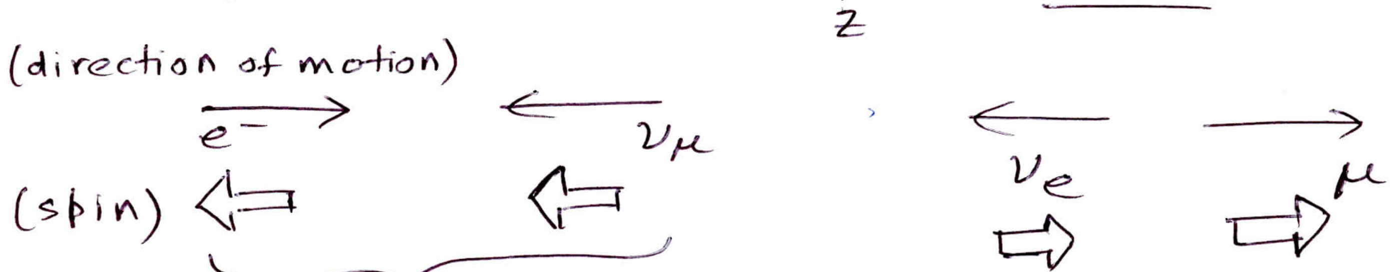
In the limit $m_\mu = 0$, $m_e = 0$, i.e., "chirality = helicity" for μ, e as well (it's already valid for ν_μ, ν_e), we have

$$(\nu_\mu)_{RH} \; (e^-)_{LH} \longrightarrow (\mu^-)_{RH} \; (\nu_e)_{LH}$$

helicity (& energy) eigenstate

$\Rightarrow$ for scattering in $\boxed{\text{forward}}$ <u>direction</u> $(\theta = 0)$: between μ^- & e^-

<u>initial</u> state $\xrightarrow{\quad}$ z <u>final</u> state



$S_z^{\text{initial}} = -1$ $\qquad$ $S_z^{\text{final}} = 1$

Since motion is along z-direction, we have $L(\text{orbital})_z^{\text{initial, final}} = 0$

$\Rightarrow$ angular momentum <u>can't</u> be conserved $\Rightarrow$ cross-section <u>vanishes</u>

– Note (i) above argument doesn't go through for $\theta \neq 0$, i.e., need to calculate!

(ii) if $m_{\mu,e} \neq 0$, then even though only L (R) chirality of e^- (μ^-) interacts, it corresponds to a combination of LH & RH helicities (since "chirality ≠ helicity" now) ⇒ above argument doesn't seem to go through ... but we expect $d\sigma/d\Omega$ for $\theta = 0 \propto m_{\mu,e}$ ...

– Back to part (2) of calculation (specific $\mathcal{M}_{fi}$): as for decay, sum over final spins and average over initial spins (i.e., probability for e^- to be in each spin state $= 1/2$) ⇒

$$\left(\tfrac{1}{2}\right)^2 \sum_{\substack{\text{initial}\\\text{spins}}} \sum_{\substack{\text{final}\\\text{spins}}} |\mathcal{M}_{fi}|^2 \left(\equiv \overline{|\mathcal{M}_{fi}|^2}\right),$$ assuming ν's also have 2 helicities/chiralities (even though massless)

$$= 2 G_F^2 \sum_{\substack{s_1, s_2\\ s_1', s_2'}} \left[\bar{u}_{\nu_e s_2}(k') \gamma^\lambda L\, u_{e s_1}(p)\right] \left[\bar{u}_{\mu s_1'}(p') \gamma_\lambda R\, u_{\nu_\mu s_2}(k)\right]$$

$$\times \left[\bar{u}_{\nu_e s_2'}(k') \gamma^\rho L\, u_{e s_1}(p)\right]^* \left[\bar{u}_{\mu s_1'}(p') \gamma_\rho R\, u_{\nu_\mu s_2}(k)\right]^*$$

spin ...

note: different dummy index than in line above

$$\equiv G_F^2/8\, (T_e)^{\lambda\rho} (T_\mu)_{\lambda\rho}$$

where $(T_e)^{\lambda\rho} = \sum_{s,s'} \left[\bar{u}_{\nu_e s'}(k')\gamma^\lambda(1-\gamma_5)u_{es}(p)\right] \times$ (spin)

$\left[\bar{u}_{\nu_e s'}(k')\gamma^\rho(1-\gamma_5)u_{es}(p)\right]^+$ same as "*"

$$(T_\mu)_{\lambda\rho} = \sum_{s,s'} \left[\bar{u}_{\mu s'}(p')\gamma_\lambda(1+\gamma_5)u_{\nu_\mu s}(k)\right] \times$$
$$\left[\bar{u}_{\mu s'}(p')\gamma_\rho(1+\gamma_5)u_{\nu_\mu s}(k)\right]^+$$

$\boxed{\text{Evaluate}}$ $T_{e,\mu}$ by analog of Eq. 7.18 of Lahiri & Pal ... or do it from "scratch":

$\boxed{2^{nd}}$ factor in $(T_\mu)_{\lambda\rho}$

$$= u^+_{\nu_\mu}\left[\gamma_\rho(1+\gamma_5)\right]^+\gamma_0 u_\mu \quad \text{(dropping s,k etc. labels)}$$

$$= \bar{u}_{\nu_\mu}\gamma^0(1+\gamma_5)\gamma_\rho^+\gamma_0 u_\mu$$

$$= \bar{u}_{\nu_\mu}(1-\gamma_5)\underbrace{\gamma^0\gamma_\rho^+\gamma_0}_{\gamma_\rho} u_\mu$$

$$= \bar{u}_{\nu_\mu}\gamma_\rho(1+\gamma_5)u_\mu$$

$$\Rightarrow (T_\mu)_{\lambda\rho} = \sum_{\text{spins}} (\bar{u}_\mu)_\alpha\left[\gamma_\lambda(1+\gamma_5)\right]_{\alpha\beta}(u_{\nu_\mu})_\beta$$

Dirac index (i.e, $1 \to 4$)

$$\times (\bar{u}_{\nu_\mu})_\omega\left[\gamma_\rho(1+\gamma_5)\right]_{\omega\sigma}(u_\mu)_\sigma$$

$$= \sum_{\text{spins}} (u_\mu)_\sigma(\bar{u}_\mu)_\alpha\left[\gamma_\lambda(1+\gamma_5)\right]_{\alpha\beta}(u_{\nu_\mu})_\beta(\bar{u}_\nu)_\omega\left[\gamma_\rho(1+\gamma_5)\right]_{\omega\sigma}$$

"signals" trace

Using $\sum_{\text{spins}} u\bar{u} = (\not{p} + m)$; $\sum_{\text{spins}} v\bar{v} = (\not{p} - m)$

we get (with $m_{\nu_\mu} = 0$)

use $(1+\gamma_5)^2 = 2(1+\gamma_5)$

$$(T_\mu)_{\lambda\rho} = \mathrm{Tr}\left[(\not{p}' + m_\mu)\gamma_\lambda(1+\gamma_5)\not{k}\gamma_\rho(1+\gamma_5)\right]$$

$$= \mathrm{Tr}\left[(\not{p}' + m_\mu)\gamma_\lambda \not{k} \gamma_\rho (1+\gamma_5)\right] \times 2$$

Use (i) $\mathrm{Tr}[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho] = 4(g_{\mu\nu}g_{\lambda\rho} + g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho})$ (Eq. A.24 of Lahiri, Pal)

(ii) $\mathrm{Tr}[\underline{\text{odd}}$ number of γ_μ's$] = 0$

and (iii) $\mathrm{Tr}[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\rho \gamma_5] = 4i\varepsilon_{\mu\nu\lambda\rho}$ (Eq. A.26)

to give

$$(T_\mu)_{\lambda\rho} = \mathrm{Tr}\left[\underbrace{\not{p}'}_{=p'^\alpha\gamma_\alpha}\gamma_\lambda \underbrace{\not{k}}_{=k^\beta\gamma_\beta}\gamma_\rho(1+\gamma_5)\right] \times 2$$

[again, m_μ term involves Tr(odd number of γ_μ's)]

$$= 8\Big[p'_\lambda k_\rho + p'_\rho k_\lambda - (p'\cdot k)g_{\lambda\rho} \leftarrow \underline{\text{symmetric}} \text{ in } \lambda \leftrightarrow \rho$$

$$+ i\varepsilon_{\alpha\lambda\beta\rho}\, p'^\alpha k^\beta \leftarrow \underline{\underline{\text{anti}}}\text{symmetric in } \lambda \leftrightarrow \rho\Big]$$

Similarly [i.e., with $\mu \to e \ldots$; $p' \to p \ldots$; $(1+\gamma_5) \to (1-\gamma_5)$ and $\rho \leftrightarrow \lambda$ (and both being raised)], we get (do it by "brute force" to make sure!)

$$(T_e)^{\lambda\rho} = 8\Big[p^\rho k'^\lambda + p^\lambda k'^\rho - (p\cdot k')g^{\lambda\rho} \leftarrow \underline{\text{symmetric}}\ldots$$

$$\underline{-}\, i\varepsilon^{\omega\rho\sigma\lambda}\, p_\omega k'_\sigma \leftarrow \underline{\underline{\text{anti}}}\ldots\Big]$$

compare sign to $(T_\mu)_{\lambda\rho}$ above

$$\left[-i\varepsilon_{\lambda}^{\text{in } (T_e)^{\lambda\rho}}\cdots \text{ vs } +i\varepsilon_{\lambda}^{\text{in } (T_\mu)_{\lambda\rho}}\cdots \text{ due to } (1-\gamma_5) \text{ vs. } (1+\gamma_5)\right]$$

$\Rightarrow$ In $\boxed{(T_e)^{\lambda\rho}(T_\mu)_{\lambda\rho}}$ only (symmetric in $\lambda \Leftrightarrow \rho$)[2] and (anti-symmetric ...)[2] survives so that

$$|\mathcal{M}_{fi}|^2 = 8G_F^2 \times \Big\{ \big[(p'\cdot k')(p\cdot k) + (p'\cdot p)(k\cdot k') - (p\cdot k')(p'\cdot k) \big] \times 2$$

from $p'_\lambda k_\rho$ in $(T_\mu)_{\lambda\rho}$ with symmetric ... of $(T_e)^{\lambda\rho}$ / $p'_\rho k_\lambda$ in $(T_\mu)_{\lambda\rho}$ with symmetric ... of $(T_e)^{\lambda\rho}$

$$+\big[-(p'\cdot k)(p\cdot k') - (p'\cdot k)(p\cdot k') + (p'\cdot k)(p\cdot k') \times 4 \leftarrow g_{\lambda\rho}g^{\lambda\rho} \big]$$

from $g_{\lambda\rho}$ in $(T_\mu)_{\lambda\rho}$ with symmetric ...

$$+\big[p_\omega k'_\sigma p'^\alpha k^\beta \times \varepsilon_{\alpha\lambda\beta\rho}\varepsilon^{\omega\rho\sigma\lambda} \big] \Big\}$$

from $+i\varepsilon\ldots$ of $(T_\mu)_{\lambda\rho}$ with $-i\varepsilon\ldots$ of $(T_e)^{\lambda\rho}$

– $(p'\cdot k)(p\cdot k')$ cancel in above $\{\ldots\}$

– Use $\varepsilon_{\alpha\lambda\beta\rho}\varepsilon^{\omega\rho\sigma\lambda} = (-\varepsilon_{\lambda\rho\beta\alpha})(-\varepsilon^{\lambda\rho\omega\sigma})$

$$= -2(g^\omega_\beta g^\sigma_\alpha - g^\omega_\alpha g^\sigma_\beta) \quad (\text{Eq. A.33})$$

in 3rd $[\ldots]$ above

to give

$$|\overline{\mathcal{M}}|^2 = G_F^2 \times 8 \times \Big\{ 2(p' \cdot k')(p \cdot k)$$

$$+ 2(p' \cdot p)(k \cdot k') \quad \leftarrow \text{from } 1^{st} [\ldots] \text{ above}$$

$$- 2(p \cdot k)(p' \cdot k') + 2(p \cdot p')(k \cdot k') \quad \leftarrow \text{from } 3^{rd} [\ldots] \ldots \Big\}$$

$$= 32\, G_F^2 \, (p \cdot p')(k' \cdot k) \quad \text{(Lorentz-invariant as expected)}$$

Simplify $|\overline{\mathcal{M}}|^2$ in CM frame :

$$p \cdot p' = E_e E_\mu - \vec{p} \cdot \vec{p}'$$

$$= \sqrt{|\vec{p}|^2 + m_e^2}\,\sqrt{|\vec{p}'|^2 + m_\mu^2} - |\vec{p}|\,|\vec{p}'| \cos\theta$$

[use $|\vec{p}| = \sqrt{\dfrac{(s+m_e^2)^2}{4s} - m_e^2} = \dfrac{s - m_e^2}{2\sqrt{s}}$ since

$m_2 = 0$ in earlier formula for $|\vec{p}_1|$

and similarly $|\vec{p}'| = (s - m_\mu^2)/(2\sqrt{s})$]

$$= \left[(s + m_e^2)(s + m_\mu^2) - (s - m_e^2)(s - m_\mu^2)\cos\theta\right] / (4s)$$

and $k \cdot k' = E_{\nu_\mu} E_{\nu_e} - \vec{k} \cdot \vec{k}' = |\vec{k}|\,|\vec{k}'|(1 - \cos\theta)$

$$= \left[(s - m_e^2)(s - m_\mu^2)(1 - \cos\theta)\right] / (4s)$$

[using $m_{\nu_{e,\mu}} = 0$ and $|\vec{k}| = |\vec{p}|$; $|\vec{k}'| = |\vec{p}'|$]

$\Rightarrow$ cross-section vanishes in forward direction ($\theta = 0$) even if $m_{\mu, e} \neq 0$ (due to $k \cdot k'$ factor in $|\overline{\mathcal{M}}|^2$ being zero for $\theta = 0$)

[If $m_{\mu, e} = 0$, then even $p \cdot p'$ factor in $|\overline{\mathcal{M}}|^2$ vanishes for $\theta = 0$] (parity invariance + angular momentum)

... vanishing was "expected" for $m_{\mu, e} = 0$ ... but no such expectation for $m_{\mu, e} \neq 0$!

— x —

Finally, plug above $|\overline{\mathcal{M}}|^2$ and $|\vec{p}|, |\vec{p}'|$ in general formula for cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{G_F^2 s}{32\pi^2} \times \left(\frac{s - m_\mu^2}{s - m_e^2} \right) \times \left[\left(1 - \frac{m_e^2}{s}\right)\left(1 - \frac{m_\mu^2}{s}\right)(1 - \cos\theta) \right] \times \left[\left(1 + \frac{m_e^2}{s}\right)\left(1 + \frac{m_\mu^2}{s}\right) - \left(1 - \frac{m_e^2}{s}\right)\left(1 - \frac{m_\mu^2}{s}\right)(1 - \cos\theta) \right]$$

Check mass dimensions: $[G_F] = -2$, $[s] = +2$ (mass dimension)

$\Rightarrow \left[\frac{d\sigma}{d\Omega} \right] = -2$, i.e., area (as expected)