(it turns out in <u>real</u> world, e.g. QED, we have <u>both</u> $\psi_{L,R}$)

- Again, $\psi_{L,R}$ don't mix (i.e., transform <u>in</u>dependently)

- Based on infinitesimal version of above LT, <u>generators</u> acting on $\psi$ (again 4-component) are

$$J^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \text{ for } \underline{\text{rotations}} \left(\text{i.e., } \underline{2} \text{ copies of 3D rotations}\right)$$

$$\& \; K^i = \frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \text{ for } \underline{\text{boosts}}$$

due to $\mp\beta$

- <u>In</u>formal HW (i.e., <u>not</u> to be submitted):

check that above $J^i, K^i$ (6 generators) satisfy usual algebra of Lorentz group:

$$[J_i, J_j] = i\varepsilon^{ijk} J_k \; ; \; [K_i, K_j] = -i\varepsilon^{ijk} J_k \; ; \; [K_i, J_j] = i\varepsilon^{ijk} K_k$$

(The 1st commutator should be familiar to all of you. In case, you are not familiar with commutators of <u>boost generators</u>, just figure out $K^i$ for boosts acting <u>on</u> $\boxed{x^\mu}$ ... and then compute commutators of those $K$'s...)

(from infinitesimal version)

(I'm sort of following Peskin, Schroeder section 3.2 here)

Step (2): We'd like to determine Lorentz invariant differential equation for $\psi$ (Dirac spinor)

For this purpose, a more useful notation for the 6 generators is obtained by introducing

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \Big\} \boxed{\text{"}\gamma^\mu\text{"}}$$

more on whether this is a Lorentz vector index in a bit

and $\boxed{S^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]}$

antisymmetric

so that $S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$ (see HW4)

are boost generators ($K^i$'s above)

and $S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$ (see HW4)

$= \varepsilon^{ijk} J^k$

as above

are rotation generators

and finite LT on $\psi$ is implemented by

$$\boxed{\Lambda_{1/2} \equiv \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}\, S^{\mu\nu}\right)}$$

real

$\omega_{\mu\nu}$ antisymmetric $\Rightarrow$ 6 real parameters (for 3 rotations + 3 boosts)

— Back to Lorentz invariant equation for $\psi$: can we use new object "$\gamma^\mu$" to construct a differential operator ($D$) such that

$\mathcal{D}\psi$ transforms (under Lorentz group) in same way as $\psi$? If yes, then equation of form $\mathcal{D}\psi = c\psi$ is Lorentz-invariant
$c$ → constant number

(Of course $\mathcal{D} = \Box$ does the job already - giving KG equation - but $\Box$ acts trivially in Dirac space, i.e., doesn't "mix-up" 4 components of $\psi$ ... whereas $\mathcal{D}$ containing $\gamma^\mu$ would be non-trivial...)

— It looks like $\mathcal{D} = \gamma^\mu \partial_\mu$ is good "candidate"... but $\gamma^\mu$ is really a constant matrix (i.e., doesn't transform under Lorentz group) — in fact, why assign "$\mu$" on $\gamma$ then? $\boxed{\text{claim}}$ is that
Actually, $\gamma^\mu \partial_\mu$ is Lorentz-invariant

because $\gamma^\mu$ does sort of "behave as vector" as follows

(i) we can show (HW4) that above $\gamma^\mu$'s satisfy

$$[\underset{4\times 4}{\gamma^\mu}, \gamma^\nu]_+ = 2g^{\mu\nu} \mathbb{1}_{4\times 4}$$

← anticommutator (in notation of Lahiri & Pal)

only (i.e., no matter exact form of $\gamma$)

(ii) Using above anticommutator, we can show (HW4)

$$[\gamma^\mu, S^{\rho\sigma}]_- = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \, \gamma^\nu$$

← commutator

$\gamma^\mu$, $S^{\rho\sigma}$: matrices in spinor space) — $(\mathcal{J}^{\rho\sigma})^\mu{}_\nu$: not matrix in spinor space where $\left(\mathcal{J}^{\mu\nu}\right)_{\alpha\beta}$ → antisymmetric (6 parameters of Lorentz transformation) correspond to

$$= i\left(\delta^{\mu}{}_{\alpha}\,\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\,\delta^{\nu}{}_{\alpha}\right)$$

[with $\left(\mathcal{J}^{\mu\nu}\right)^{\alpha}{}_{\beta} = g^{\alpha\delta}\left(\mathcal{J}^{\mu\nu}\right)_{\delta\beta}$]

use $g^{\cdot\cdot}$ to raise/lower indices as usual

What is this <u>new object</u> $\boxed{\mathcal{J}^{\mu\nu}}$? ... simply generator of LT on 4-vector, i.e.,

$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ with
$\Lambda^{\mu}{}_{\nu} \approx \delta^{\mu}_{\nu} - \frac{i}{2}\,\omega_{\rho\sigma}\left(\mathcal{J}^{\rho\sigma}\right)^{\mu}{}_{\nu}$

[check: (a) choose $\omega_{10} = -\omega_{01} = \beta$ to be only non-zero entries so that

$$\Lambda^{\mu}{}_{\nu} \approx \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & & 1 \end{pmatrix}$$

i.e., infinitesimal version of boost in $x$ direction:
$x' = \gamma(x - \beta t)$; ↳ not matrix!
$t' = \gamma(t - \beta x)$;
$y' = y$ ; $z' = z$

Similarly, $\omega_{12} = -\omega_{21} = \theta$

gives rotation about z-axis
i.e., $\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & +\theta & 1 & \\ 0 & 0 & 0 & 1 \end{pmatrix}$ ]

— Using above $[\gamma^{\mu}, S^{\rho\sigma}] = \ldots$, we can show that

$$\left(1+\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1-\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)$$

$$= \left[\delta^\mu_\nu - \tfrac{i}{2}\omega_{\rho\sigma}\left(\mathcal{J}^{\rho\sigma}\right)^\mu{}_\nu\right]\gamma^\nu$$

... exponentiating which gives — matrix in vector space

$$\boxed{\Lambda^{-1}_{1/2}\,\gamma^\mu\,\Lambda_{1/2} = \Lambda^\mu{}_\nu\,\gamma^\nu}$$

matrices in spinor space

i.e., $\gamma^\mu$ has dual role: we can "rotate" its spinor index (using $\Lambda_{1/2}$) or rotate "$\mu$"-index (by LT/on $x^\mu$) ... and these two operations have same result

(like $\sigma^i$ which is matrix in spinor space, but we can act on its "$i$" index by a $3\times 3$ rotation matrix which acts on $x^i$)

— Real meaning of all this is $\gamma^\mu\partial_\mu$ is Lorentz invariant operator:

$$\underbrace{\gamma^\mu}_{\text{no "prime" on } \gamma}\ \overbrace{\partial^{(1)}_\mu}^{\frac{\partial}{\partial x^{\mu(1)}}}\ \psi^{(1)}(x') \quad (\text{i.e., } \gamma^\mu\partial_\mu\psi \text{ in new frame})$$

$$= \gamma^\mu\,\partial_\mu'\left[\underset{(4\times 4)}{\Lambda_{1/2}}\ \underset{(4\times 1)}{\psi(x)}\right]$$

trivial in spinor space

$$= \gamma^\mu\frac{\partial}{\partial x^{\mu'}}\,\Lambda_{1/2}\,\psi\Big(\underbrace{(\Lambda^{-1})^\nu{}_\alpha\, x'^\alpha}_{=x}\Big)$$

i.e., $d/dx\, f(ax) = a \times \left(d/dy\, f(y)\right)\big|_{y=ax}$ chain rule

$$= \gamma^\mu \Lambda_{1/2} \left(\Lambda^{-1}\right)^\nu{}_\mu \underbrace{\left(\partial_\nu \psi\right)\left(\Lambda^{-1}x'\right)}_{\text{i.e., } \frac{\partial}{\partial y^\nu}\psi(y)\big|_{y=\Lambda^{-1}x'}}$$

insert $\Lambda_{1/2}\Lambda_{1/2}^{-1}$

$$= \Lambda_{1/2} \underbrace{\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2}}_{\Lambda^\mu{}_\sigma \gamma^\sigma \text{ from above}} \left(\Lambda^{-1}\right)^\nu{}_\mu \left(\partial_\nu \psi\right)\left(\Lambda^{-1}x'\right)$$

$$= \Lambda_{1/2} \underbrace{\delta^\nu{}_\sigma}_{\text{from } \Lambda^\mu{}_\sigma (\Lambda^{-1})^\nu{}_\mu} \gamma^\sigma \left(\partial_\nu \psi\right) \underbrace{\left(\Lambda^{-1}x'\right)}_{=x}$$

$$= \Lambda_{1/2} \left(\gamma^\nu \partial_\nu \psi\right)(x)$$

i.e., $\gamma^\mu \partial_\mu \psi$ transforms like $\psi$ ... (as desired)
[again, $\psi'(x') = \Lambda_{1/2}\psi(x)$]

— Thus, $i\gamma^\mu \partial_\mu \psi = m\psi$ is Lorentz-invariant (Dirac equation)

— Why "$i$", "$m$" in Dirac equation?
"2"

It better be that (DE) $\Rightarrow$ KG equation since latter follows simply from energy-momentum relation for particle (no matter its spin) — check it:

$$\left(-i\gamma^\mu \partial_\mu - m\right) \underbrace{\left(i\gamma^\nu \partial_\nu - m\right)\psi}_{=0 \text{ by DE}} = 0$$

$$\Rightarrow \left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi + m^2\right)\psi = \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu + m^2\right)\psi$$
$$= \left(\Box + m^2\right)\psi$$

— Then, why do we need DE? Because 1st order equation contains additional information ... which gives spin-½:

— Solutions for $\psi$ must be $\sim \underbrace{e^{\mp i p^\mu x_\mu}}_{\text{KG equation solutions}}$

× 4-component (constant) spinor (like Fourier coefficient)

with $p^0 = \sqrt{|\vec{p}|^2 + m^2}$

— In rest frame ($p^0 = m$), we get

$$\left(m\gamma^0 - m\right)\boxed{u} = 0 \quad , \text{ i.e., } \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u = 0$$

spinor

choose $p^0 > 0$

$\Rightarrow u \sim \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ → any 2-component spinor

→ same!

e.g. choose basis for $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow$ $\boxed{\text{two}}$ $u$'s

are eigenstates of $J^3 = \frac{1}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$, i.e.,

generator of rotation about $z$-axis for Dirac spinor

$\Rightarrow$ spin up & down (i.e., two) states

— If we used only KG equations, then all 4 components of $u$ are independent (i.e., DE "mixes up" ...)

$\boxed{\text{i.e., } \underline{\text{not}} \text{ eigenstates of } J^3 \text{ (in general)}}$

— Of course, $p^0 < 0$ gives $\boxed{v} \sim \begin{pmatrix} \xi \\ \theta\xi \end{pmatrix}$

i.e., another set of spin up & down

... which are associated with

anti-particles

— Finally, other "representations" of $\gamma^\mu$'s :
defining relation is $[\gamma^\mu, \gamma^\nu]_\oplus = 2 g^{\mu\nu} \mathbb{1}_{4\times4}$

For any such $\gamma$'s, define $S^{\mu\nu} = i/4 [\gamma^\mu, \gamma^\nu]_\ominus$

(as we did above for specific choice of $\gamma$'s)

We can show that $\boxed{S^{0i}}$ and $\boxed{J^i \equiv \varepsilon^{ijk} S^{jk}}$

will be boost & rotation generators (HW4) etc

Further, more, we can show $[\gamma^\mu, S^{\rho\sigma}] = \dots$ (which anyway) any choice of $\gamma$

we (already) showed above using only $[\gamma^\mu, \gamma^\nu]_\oplus = \dots$

(i.e., we didn't use specific form of $\gamma$ to show it)