

**PHYS 624 (Fall 2010):
Advanced Quantum Mechanics
Homeworks**

1 Homework 1 (Classical Field Theory), due Wednesday, September 15

1.1 Electromagnetic Field

The idea behind these problems is to “re-derive” some of the known results in electromagnetism using the classical field theory approach, i.e., with the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \quad (1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2)$$

and identifying the electric and magnetic fields as

$$E^i = -F^{0i}, \quad (3)$$

$$\epsilon^{ijk}B^k = -F^{ij} \quad (4)$$

For example, we already showed in lecture that Maxwell’s equations are simply the Euler-Lagrange equations.

1.1.1 Energy-momentum

Based on Noether’s theorem, construct the energy-momentum tensor for classical electromagnetism from the above Lagrangian.

Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad (5)$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu \quad (6)$$

leads to an energy-momentum tensor \hat{T} that is symmetric and yields the *standard* (i.e., known *without* using field theory) formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}(E^2 + B^2), \quad (7)$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} \quad (8)$$

1.1.2 Subtlety with going to Hamiltonian formalism

Exercises 2.4 and 2.5 of Lahiri and Pal.

Due to this subtlety, we will *not* quantize electromagnetic field to begin with (even though historically it was the first QFT). We will return to this issue when we quantize the electromagnetic field later in the course.

1.2 Real, free scalar/Klein-Gordon Field

This is the simplest classical field theory and so the *first* one that we will quantize. For the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2, \quad (9)$$

where ϕ is a real-valued field,

- (i) show that the Euler-Lagrange equation is the Klein-Gordon equation for the *field* ϕ .
- (ii) Find the momentum conjugate to $\phi(x)$, denoted by $\Pi(x)$.
- (iii) Use $\Pi(x)$ to calculate the Hamiltonian density, \mathcal{H} .
- (iv) Based on Noether's theorem, calculate the stress-energy tensor, T_ν^μ , of this field and the conserved charges associated with time and spatial translations, i.e., the energy-momentum, P^μ , of this field.
- (v) Using the Euler-Lagrange (i.e., Klein-Gordon) equation, show that $\partial_\mu T_\nu^\mu = 0$ for this field. (Of course, this result was expected from Noether's theorem.)
- (vi) Finally, show that P^0 that you calculated above in part (iv) is the same as the total Hamiltonian, i.e., spatial integral of \mathcal{H} which you calculated above in part (iii).

We will determine eigenstates/values of this (total) Hamiltonian when we quantize the field.

And, P^i can be interpreted as the (physical) momentum carried by the field (not to be confused with canonical momentum!). This P_i will be used in interpreting the eigenstates of the Hamiltonian of the quantized scalar field.

1.3 Complex scalar/Klein Gordon field coupled to electromagnetism (Scalar electrodynamics)

Exercises 2.9 (b) and (c) of Lahiri and Pal. Neglect the potential term, $V(\phi^\dagger \phi)$, given in the Lagrangian in exercise 2.3 of Lahiri and Pal for these problems.

The *free* complex Klein-Gordon field was discussed in lecture. In particular, it was already shown that the Euler-Lagrange equation is the Klein-Gordon equation (exercise 2.3 of Lahiri and Pal) and the conserved current corresponding to the transformation $\phi \rightarrow e^{i\alpha} \phi$ was already calculated [exercise 2.9 (a) of Lahiri and Pal] so that that there is *no* need to do it

again here. This field is a simple generalization of the case of a real field so that it will be the *second* field to be quantized.

The purpose of exercises 2.9 (b) and (c) in Lahiri and Pal is to study the *addition* of an interaction of this field with the electromagnetic field. We will return to quantization of this theory later in the course.

1.4 Scale invariance

Exercise 2.10 of Lahiri and Pal.

The transformations involve a simultaneous re-scaling of the coordinates and the fields, hence the name “scale invariance” given to this symmetry.

2 Homework 2 (Quantization of Scalar Field: part I), due Friday, September 24

2.1 Toy model for radiation field

(i) Exercise 2.7 of Lahiri and Pal.

(ii) Before doing the above exercise, as a “*zeroth*” step, show that the Euler-Lagrangian equation for this Lagrangian is the usual wave equation.

As mentioned in class, historically the electromagnetic field was the first one to be quantized. However, as we saw in homework problem 1.1.2, there are subtleties with going to Hamiltonian formalism for electromagnetic field (related to gauge invariance). We will return to this issue later in the course, but for now, a sort of “toy model” for the electromagnetic field will suffice. That is the motivation behind this exercise, i.e., it serves as a “warm-up”.

Note also that the above exercise involves imposing commutation relations for *Fourier coefficients* of the field (directly interpreting them as creation/annihilation operators), rather than for the fields themselves (in lecture, we will follow the latter approach) – of course, the two ways are related.

2.2 Commutation relations for a , a^\dagger from those for ϕ , Π

Exercise 3.4 of Lahiri and Pal.

Note that in lecture, the “reverse” procedure was preformed.

2.3 Momentum operator in terms of a , a^\dagger

(i) Just like we did for the Hamiltonian in lecture, rewrite the 3-momentum operator for a classical real scalar field that you calculated in problem 1.2 (using Noether’s theorem, i.e., as the conserved charge associated with spatial translations) in terms of the creation and annihilation operators for the quantized real scalar field.

You should find that, unlike for Hamiltonian, normal ordering is not required.

(ii) Exercise 3.5 b) of Lahiri and Pal.

2.4 More examples of dealing with annihilation and creation operators and their commutation relations

(i) Exercise 3.3 of Lahiri and Pal

(ii) Number operator: exercise 3.6 of Lahiri and Pal.

We will return to this number operator when we discuss complex scalar field.

3 Homework 3 (Quantization of Scalar Field: part II), due Monday, October 4

3.1 $U(1)$ symmetry for complex scalar

3.1.1 Two real scalars with *different* masses

Consider the theory of two real scalar fields, but with different masses, i.e.,

$$\mathcal{L} = \sum_{A=1}^2 \left[\frac{1}{2} (\partial^\mu \phi_A) (\partial_\mu \phi_A) - \frac{1}{2} m_A^2 \phi_A^2 \right] \quad (10)$$

with $m_1 \neq m_2$.

(i) Rewrite this Lagrangian in terms of a (single) complex scalar field,

$$\phi(x) = \frac{1}{\sqrt{2}} \left[\phi_1(x) + i\phi_2(x) \right] \quad (11)$$

(ii) Determine if the Lagrangian for the complex scalar field is invariant under the transformation

$$\phi \rightarrow e^{-iq\theta} \phi \quad (12)$$

(iii) Is there a motivation then for dealing with the complex scalar field instead of the two real scalar fields (compare to the case studied in lecture where the two real scalar fields have *same* mass)?

3.1.2 Using $a_{1,2}$ instead of a, \hat{a}

(i) Exercise 3.7 of Lahiri and Pal.

Obviously, one cannot then have a simple “particle and anti-particle” interpretation if we think in terms of $a_{1,2}$: compare to the case of writing the number operator in terms of the particular *combination* of $a_{1,2}$, namely, a, \hat{a} (as discussed in lecture) - that is the motivation for using the combinations a and \hat{a} .

3.2 Two complex scalars

Consider the case of *two* complex scalar fields with the *same* mass, labelled as $\phi_a(x)$ with $a = 1, 2$.

(i) Show that there are now *four* conserved charges, one given by the generalization of the case of single complex scalar, and the other three given by

$$Q^i = \int d^3x \frac{i}{2} \left[\phi_a^* (\sigma^i)_{ab} \pi_b^* - \pi_a (\sigma^i)_{ab} \phi_b \right] \quad (13)$$

where σ^i are the Pauli sigma matrices.

(ii) Show that the above three charges have the commutation relations of angular momentum, i.e., $[SU(2)]$.

Although we probably will not discuss it in this course (but it will be covered in Phys 751/752), the above theory is similar to that of the Higgs field in the standard model of particle physics.

3.3 Causality

As discussed in class, the equal-time commutation relations are not manifestly covariant (even though the equation of motion, the Klein-Gordon equation in the case of scalar field, is). So, one might worry whether that leads to any “problems”, for example whether causality is violated.

As a “diagnostic” of causality violation, we can consider the commutator of two fields at two different space-time points. If this commutator vanishes, then one measurement of the field cannot affect the other. Of course, such commutators vanish *at equal time* by the canonical commutation relations so that one is left to compute them for *non*-equal times.

3.3.1 Complex scalar

We discussed causality for a *real* scalar in lecture. Repeat here the same arguments for *complex* scalar as follows (even if some of the steps might be similar to those done in lecture for a real scalar, you should work them out again here just to make sure you understand them).

(i) Using the expansion of ϕ in terms of creation and annihilation operators, calculate $[\phi(x), \phi^\dagger(y)]$ in general, i.e., not assuming equal times. You should find two terms in the result (there is *no* need to simplify by combining them).

(ii) Verify that the above two terms (rather trivially) cancel for equal times so that the commutator indeed vanishes (as expected).

(iii) Show that each of the above two terms in the commutator is *separately* Lorentz-invariant (and thus the entire commutator is Lorentz invariant).

(**Hint:** rewrite the $\int d^3p$ as a $\int d^4p$.)

(iv) Using the Lorentz invariance of the commutator and its vanishing for equal-times, show that, for space-like separation, i.e., $(x - y)^2 < 0$, the commutator vanishes even for *unequal* times. Thus, causality is indeed not violated.

(**Hint:** use a suitable reference frame to evaluate the commutator.)

(v) Show that for time-like separation, i.e., $(x - y)^2 > 0$, the above manipulation does *not* go through so that the commutator cannot be made to vanish – of course, we would like it not to so that two measurements which are *inside* each others' light cones do affect each other!

(vi) Interpret the vanishing of commutator for space-like separation (i.e., preservation of causality) in terms of particle and *anti*-particle propagation between the two space-time points (being careful with the order of the two points). For this purpose, use the fact that this commutator is a *c*-number so that it be written as $\langle 0 | [\phi(x)\phi^\dagger(y)] | 0 \rangle$ and $\phi(x)|0\rangle$ can be interpreted as particle localized at x (and similarly for ϕ^\dagger).

You should then be able to conclude that causality *requires* existence of anti-particles with same mass as the particle.

3.4 Showing that Feynman propagator *is* a Green's function

Exercise 3.9 of Lahiri and Pal.

3.5 Propagator for *complex* scalar field

Exercise 3.10 of Lahiri and Pal.

In lecture, we discussed various aspects of propagator for *real* scalar field and so here the goal is to do it for complex scalar field. The Green's function for (classical) complex scalar field is same as that for real scalar field (think about why this is so). However, the relation between Green's function and quantized field (and thus particle propagation) is modified - that's the motivation behind this problem.

4 Homework 4 (Dirac equation), due Wednesday, October 13

This homework involves lots of (useful) algebra: some of the relations derived here were already used in lecture while others will be used later on in the course. Another motivation for this homework is to familiarize you with Dirac γ -matrices.

All problems here (other than 4.1.1 and 4.1.2) are to be solved in a Dirac γ -matrix representation-independent way, i.e., do *not* assume Eq. (14) in the other problems.

4.1 Weyl/chiral representation for the Dirac γ -matrices

4.1.1 Anti-commutation relations

Show that

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\end{aligned}\tag{14}$$

satisfy

$$\left[\gamma^\mu, \gamma^\nu\right]_+ \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1}\tag{15}$$

4.1.2 Boost and rotation generators

Define

$$S^{\mu\nu} = \frac{i}{4} \left[\gamma^\mu, \gamma^\nu\right]_- \equiv \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\tag{16}$$

and

$$\begin{aligned}K^i &= S^{0i} \\ J^i &= \frac{1}{2} \epsilon^{ijk} S^{ij}\end{aligned}\tag{17}$$

where $\epsilon^{123} = 1$ etc.

Show that for the Weyl/chiral representation of Dirac γ -matrices in Eq. (14) above,

$$\begin{aligned}J^i &= \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \\ K^i &= -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\end{aligned}\tag{18}$$

Using an analogy with 3-dimensional (3d), i.e., spatial, rotations, we argued in lecture that the above *specific* forms of J^i and K^i implement (infinitesimal) 4D Lorentz transformations, i.e., rotations and boosts, on four-component (Dirac) spinors.

4.2 General representation of Dirac γ -matrices

Consider a general representation of the Dirac γ -matrices, i.e., 4×4 matrices satisfying Eq. (15).

Define $S^{\mu\nu}$ as in Eq. (16) and K^i, J^i as in Eq. (17) for these matrices as well.

4.2.1 Lorentz group algebra

Show that these general J^i and K^i satisfy the algebra of the Lorentz group, i.e.,

$$\begin{aligned} [J^i, J^j]_- &= i\epsilon^{ijk} J^k \\ [K^i, K^j]_- &= -i\epsilon^{ijk} J^k \\ [K^i, J^j]_- &= i\epsilon^{ijk} K^k \end{aligned} \quad (19)$$

As mentioned above, in lecture, we derived the form of the (infinitesimal) 4D Lorentz transformations on Dirac spinors (and thus the generators of the Lorentz group) in chiral/Weyl representation, given in Eq. (18), *without* even defining the corresponding Dirac γ -matrices (let alone using their anti-commutation relations). It is easy to check “directly” – again *without* going through the Dirac γ -matrices at all – that the J^i , K^i in Eq. (18) do indeed satisfy the *algebra* of the Lorentz group (we already “knew” they would do so because of the derivation based on the 3d analogy). So, the goal here is to show it for a *general* representation of Dirac γ -matrices.

4.2.2 Meaning of “ μ ” index on γ^μ

Show that

$$[\gamma^\mu, S^{\rho\sigma}]_- = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \gamma^\nu \quad (20)$$

where

$$(\mathcal{J}^{\rho\sigma})_{\mu\nu} = i(\delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu) \text{ and} \quad (21)$$

$$(\mathcal{J}^{\rho\sigma})^\mu{}_\nu = g^{\mu\alpha} (\mathcal{J}^{\rho\sigma})_{\alpha\nu} \quad (22)$$

are the generators of Lorentz transformations on 4-vectors, for example,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (23)$$

with

$$\Lambda^\mu{}_\nu \approx \delta^\mu_\nu - \frac{i}{2} \omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu \quad (24)$$

and (anti-symmetric) $\omega_{\mu\nu}$'s parametrizing the Lorentz transformation.

The above relations were used to show in lecture that $\gamma^\mu \partial_\mu$ is a Lorentz-invariant operator for a *general* representation of the Dirac γ -matrices..

4.2.3 Chirality projection operator

Show that

$$[\gamma^5, S^{\mu\nu}]_- = 0 \quad (25)$$

where

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (26)$$

The above relation was used in lecture to determine the chirality projection operators in a *general* representation of the Dirac γ -matrices.

4.3 Lorentz transformation of spinors

4.3.1 Modified spinor

Exercise 4.6 of Lahiri and Pal.

We will use this relation in the study of a discrete transformation called charge conjugation.

4.3.2 Bilinears

(i) Exercise 4.5 of Lahiri and Pal.

(ii) Show also that $\bar{\psi}\psi$ transforms as a scalar under Lorentz transformations.

We will use these properties in constructing a Lagrangian for Dirac field.

4.4 Gordon identity

Exercise 4.9 of Lahiri and Pal.

We will use this identity when we discuss electromagnetic form factors.

4.5 Completeness of spinors

(i) Exercise 4.8 of Lahiri and Pal

These normalization equations for spinors might be useful in the next problem.

(ii) Exercise 4.10 of Lahiri and Pal.

These completeness relations will be used in calculations of processes in quantum electrodynamics (QED).

4.6 Helicity and chirality projection operators for massless fermions

Exercise 4.19 of Lahiri and Pal (it suffices to show that the two projection operators above act identically on spinors which are solutions of Dirac equation).

Again, we showed that “helicity = chirality” in lecture using Weyl/chiral representation of the Dirac γ -matrices and for momentum along z -direction. The goal of this problem is to show it for a *general* representation and arbitrary direction of momentum.

5 Homework 5 (Quantization of Dirac field), due Friday, October 22

5.1 Anti-commutators of Dirac fields

Exercise 4.25 of Lahiri and Pal.

Note that you are supposed to check *all* (i.e., even vanishing) anti-commutators.

The precise relation for the non-vanishing anti-commutator is

$$\left[\psi_\alpha(t, \mathbf{x}), \psi_\beta^\dagger(t, \mathbf{y}) \right]_+ = \delta_{\alpha\beta} \delta^3(\mathbf{x} - \mathbf{y}), \quad (27)$$

where α, β are Dirac spinor indices.

Note that these anti-commutators are to be calculated at *equal* times (as usual).

In lecture we used the anti-commutation relations for f, \hat{f} etc. to determine the Hamiltonian of the fermion/Dirac field, i.e., we did it *without* using the anti-commutation relations for the fields themselves.

5.2 Momentum operator

(i) Calculate the conserved charge of the (classical) fermion/Dirac field associated with spatial translation, i.e., (linear) momentum of the fermion/Dirac field.

(ii) Rewrite the momentum of the fermion field that you calculated above in terms of the creation/annihilation operators for fermionic particles (and anti-particles), i.e., determine the momentum of the *quantized* fermion/Dirac field.

(iii) Calculate the momentum of the state $f_s^\dagger(\mathbf{p})|0\rangle$.

The momentum of this state, along with its energy and spin that we calculated in lecture, is crucial for the interpretation of this state as a single fermionic *particle*.

(iv) Is the momentum of the single *anti*-particle state, $\hat{f}_s^\dagger(\mathbf{p})|0\rangle$, where $\hat{f}_s^\dagger(\mathbf{p})$ is the coefficient of $v_s(\mathbf{p})e^{+i\mathbf{p}\cdot\mathbf{x}}$ in $\psi(x)$, given by \mathbf{p} (i.e., *opposite* to that of the v_s -spinor) or $-\mathbf{p}$ (i.e., same as that of the spinor)?

We already showed in lecture that energy and the helicity of the single anti-particle state are *opposite* to that of the corresponding spinor. These (three) “reversals of sign” agree with the prediction of the Dirac hole theory (for the wavefunction), i.e., a positron is the “absence” (from an otherwise filled “sea”) of a negative-energy electron (i.e., v_s -spinor) so that the positron’s properties are *opposite* to those of the (missing) electron.

5.3 Dirac propagator

5.3.1 Formula for Green’s function

Excercise 4.29 of Lahiri and Pal.

5.3.2 Relating Green's function to fields

Exercise 4.30 of Lahiri and Pal.

Compare the sign in definition of time-ordering to the scalar case (which is related to the use of *anti*-commutation relations for fermion vs. commutation for scalar). These signs will be relevant when we use these propagators as part of “Feynman rules” in calculations of amplitudes for various processes.

5.4 Parity transformation of Dirac and scalar fields

5.4.1 Fermion and anti-fermion states have opposite (intrinsic) parity

(i) Exercise 10.3 of Lahiri and Pal.

You can use the Weyl/chiral representation in this problem (instead of Dirac-Pauli as suggested in the book). Or, even better yet, you can try to prove the above relation in a representation-*independent* way.

The above result will be useful in the problem below.

(ii) Exercise 10.4 of Lahiri and Pal.

The above result will be useful in determining selection rules in the problem below.

5.4.2 “Selection rules” based on parity invariance, angular momentum conservation and Bose-Einstein statistics

The conservation of angular momentum and Bose-Einstein statistics (for integer-spin particles) always apply, whereas parity invariance can be violated in certain theories. However, in this problem, we will *assume* that all (unspecified) interactions are parity invariant. Even though we have not yet developed the formalism for interactions in quantum field theory, just the above assumption will be sufficient for derivation of selection rules. We will return later to the issue of how to determine whether *specific* interactions are parity-invariant or not.

The parity of a two-particle state with orbital angular momentum L (in the center-of-mass frame) can be shown to be

$$P_{two-particle} = \eta_1 \eta_2 (-1)^L \quad (28)$$

where $\eta_{1,2}$ are intrinsic parities of the two particles (which have nothing to do with wavefunction in space) and the $(-1)^L$ part comes from transformation of the wavefunction in space under the parity operation, i.e.,

$$\begin{aligned} \mathbf{x} &\rightarrow -\mathbf{x} \\ t &\rightarrow +t \end{aligned} \quad (29)$$

First, consider a spin-0 boson (a scalar particle, denoted by ϕ) of intrinsic parity +1 (i.e., even) decaying into a spin-1/2 fermion (f) and its antifermion (\bar{f}):

$$\phi \rightarrow f \bar{f} \quad (30)$$

(i) Based on parity invariance, determine the allowed values of orbital angular momentum (L) of the final state fermion and antifermion.

(ii) Using the above result and angular momentum conservation for this decay, show that the fermion and antifermion must be in a $L = 1$ state (i.e., p -wave).

Next, consider the annihilation of a spin-1/2 fermion and its antifermion into a pair of *identical* scalars of even parity:

$$f\bar{f} \rightarrow \phi\phi \quad (31)$$

(iii) Using Bose statistics, determine the allowed values of L of the final state ($\phi\phi$).

(iv) Using the above result and parity invariance, determine the allowed value of L of the initial state ($f\bar{f}$).

(v) Using above results and angular momentum conservation, show that the initial state ($f\bar{f}$) must have total spin (S) of 1.

6 Homework 6 (Formalism for interactions: restrictions on Lagrangian and Wick's theorem), due Friday, October 29

6.1 Restrictions on interaction Lagrangian

In all parts of this problem, ϕ is a *real* scalar.

6.1.1 Hermiticity

(i) Show that for the Lagrangian:

$$\mathcal{L}_{int} = h\phi\bar{\psi}\psi \quad (32)$$

to be hermitian, the coupling constant h has to be real.

(ii) Show that for the Lagrangian:

$$\mathcal{L}_{int} = h\phi\bar{\psi}\gamma_5\psi \quad (33)$$

to be hermitian, the coupling constant h has to be purely imaginary.

(iii) (a) Determine if the Lagrangian

$$\mathcal{L}_{int} = h\phi\bar{\psi}L\psi, \quad (34)$$

[where $L = 1/2(1 - \gamma_5)$] is hermitian or not.

(b) If it is not hermitian, what are the terms which should be added to the Lagrangian to make it so.

6.1.2 Lorentz invariance

You can simply “borrow” any relevant previous results, i.e., without deriving them again, on how fermion bilinears transform under Lorentz group.

(i) Given how ψ transforms under a *continuous* Lorentz transformation, determine how $\bar{\psi}\gamma_5\psi$ transforms (in particular does it transform just like a scalar i.e., same as $\bar{\psi}\psi$)?

(ii) Is the interaction term in Lagrangian (mentioned in lecture) given by

$$\mathcal{L}_{int} = (\bar{\psi}\gamma_5\psi) (\bar{\psi}\gamma_5\psi) \quad (35)$$

then Lorentz-invariant?

(iii) Repeat part (i) above for $\bar{\psi}\gamma^\mu\gamma_5\psi$ (compare to how $\bar{\psi}\gamma^\mu\psi$ transforms).

(iv) Is the interaction term in Lagrangian (mentioned in lecture) given by

$$\mathcal{L}_{int} = (\bar{\psi}\gamma^\mu\gamma_5\psi) (\bar{\psi}\gamma_\mu\gamma_5\psi) \quad (36)$$

then Lorentz-invariant?

6.1.3 Renormalizability

As mentioned in lecture, the coupling constants, i.e., coefficients of interactions terms in Lagrangian, should have non-negative mass dimension – otherwise, the theory lose predictive its power.

(i) Based on the “kinetic” term in the Dirac Lagrangian, i.e., the one containing spacetime derivatives, determine the mass dimension of Dirac/fermionic field.

(ii) Verify that the mass dimension of “mass”, i.e., $\bar{\psi}\psi$, term for the *field* in the Dirac Lagrangian is indeed +1 so that labeling that coefficient as “ m ” makes sense.

Of course, we showed that the (anti-)particles obtained by quantizing this theory do have mass m (in the usual sense).

(iii) Determine mass dimension of the Yukawa interaction, i.e., $\psi^2\phi$ (note that the details of Lorentz structure do not matter for this purpose and hence have been ignored here).

(iv) Repeat part (iii) above for the 4-fermion interaction, i.e., ψ^4 .

You should find that the mass dimension of the 4-fermion interaction is negative, i.e., it is *non-renormalizable*, so that it has to be treated as an “effective” (not a fundamental) interaction.

6.1.4 Parity invariance

Note that, unlike Lorentz invariance and hermiticity, not all interactions have to respect parity invariance.

(i) We showed in class that the term $\bar{\psi}\psi$ (i.e., mass term in Dirac Lagrangian) is parity-invariant for a certain transformation on ψ . Determine the transformation of $\bar{\psi}\gamma_5\psi$ under the same transformation of ψ .

(ii) Repeat part (i) above for $\bar{\psi}\gamma^\mu\gamma^5\psi$.

Based on above results about action of continuous Lorentz and parity transformations on the fermion bilinears, it should be clear what is meant by “pseudo” when we denote $\bar{\psi}\gamma^5\psi$ and $\bar{\psi}\gamma^\mu\gamma^5\psi$ by pseudo-scalar and pseudo-vector, respectively.

(iii) Determine the condition to be satisfied by h for the Lagrangian in Eq. (34) (made suitably hermitian) to be parity-invariant, assuming that intrinsic parity of ϕ is even.

6.2 Wick’s theorem

Exercise 5.3 of Lahiri and Pal.

In lecture, we showed a similar result for scalars. Here, the idea is to do it for fermions (keeping track of commutation relations – for scalar – vs. anti-commutation – for fermion).

7 Homework 7 (Formalism for interactions: Feynman diagrams), due Wednesday, November 10

7.1 Drawing Feynman diagrams

The interaction for this problem is the Yukawa theory discussed in lecture and in chapter 6 of Lahiri and Pal.

You should label each particle by its momentum, especially since these processes involve at least two identical particles (among the *total* of four initial and final state particles).

[**General hint:** as discussed in lecture, first determine what is the *minimum* number (which need not be “1” in general) of \mathcal{H}_{int} which will give just the right number of field operators to annihilate the initial state and create the final state.

Note that, even at a given order in \mathcal{H}_{int} , there can be *more than one* Feynman diagrams, for example, due to the different possibilities, i.e., space-time points, for annihilation/creation of initial/final state particles.

Finally, the leading-order diagram might be a tree or loop one.]

7.1.1 Scalar-fermion scattering

Draw the Feynman diagram(s), i.e., there *might* be more than one, at *lowest* order in which scalar-fermion scattering $Be^- \rightarrow Be^-$ takes place (there is no need to show higher-order diagrams).

Show them in the form of Figs. 6.1-6.3 of Lahiri and Pal, i.e., (as discussed in lecture) do *not* show other diagrams which are obtained by simple *permutations* of space-time points.

The diagrams for scalar-*antifermion* scattering will be same as above (of course with e^- replaced by e^+).

7.1.2 Fermion-*antifermion* scattering

As in above problem, but for $e^-e^+ \rightarrow e^-e^+$.

Note that this process is *different* than fermion-fermion scattering, $e^-e^- \rightarrow e^-e^-$ discussed in lecture and in Lahiri and Pal.

7.1.3 Annihilation

(i) Two scalars: as in above problems, but for $BB \rightarrow e^-e^+$.

(ii) Fermion-antifermion: as in above problems, but for $e^-e^+ \rightarrow BB$.

7.1.4 Scalar-scalar scattering

Exercise 6.1 of Lahiri and Pal.

7.1.5 All possible $2 \rightarrow 2$ processes

Are there any other such processes that we missed above (after including fermion-fermion scattering – and antifermion-antifermion scattering which will be the same – that we discussed in lecture)?

7.2 Contraction of field operators with external states

Exercise 6.3 of Lahiri and Pal.

These results are used in evaluation of S -matrix elements.

7.3 Expression for S -matrix element

Exercise 6.4 of Lahiri and Pal.

Note that this problem appears in Lahiri and Pal *before* the discussion of Feynman rules, implying that you are supposed to find the S -matrix element (again, in the form of Eq. 6.30, for example) by “brute force” – I would advise you to do follow this suggestion so that you really understand where the Feynman rules come from.

7.4 *Identical* scalar particles

(i) Calculate the analog of Eq. 6.42 of Lahiri and Pal for *real* scalar particles, i.e., $a(\mathbf{k})a(\mathbf{k}')a^\dagger(\mathbf{p}_2)a^\dagger(\mathbf{p}_1)|0\rangle$.

Such an expression would be relevant for the calculation of S -matrix element for $BB \rightarrow BB$, i.e., scalar-scalar scattering (just like Eq. 6.42 of Lahiri and Pal came up during the calculation of fermion-fermion scattering).

(ii) Compare to the result for fermions, especially relative sign between *two* terms in the evaluation of above expression. Based on this comparison, will there be a negative sign between two Feynman diagrams related by exchange of final state real scalar particles? Is this result expected based on spin-statistics theorem?

8 Homework 8 [Specific calculations of S -matrix elements (I)], due Monday, November 22

8.1 Simplifying spinors

Exercises (i) 7.1 and (ii) 7.2 of Lahiri and Pal.

These relations will be used in the calculation of decay widths and scattering cross-sections.

8.2 Decay of scalar particle into electron positron pair

8.2.1 Specific fermion helicities

Consider the limit of fermion (“electron”) being massless ($m = 0$) in this problem.

(i) In the rest frame of the scalar particle, calculate the decay width into each of the fermion helicity eigenstates.

[Hint: Since helicity projection operator is identical to chirality projection operator for massless fermion, (as discussed in class) one can simply insert L, R projection operators in the spin sum in Eq. 7.12 of Lahiri and Pal.

More explicitly, for decay into *right-handed* (RH) electron, evaluate $\sum_{s, s'} |\overline{R} u_s(\mathbf{p}) v_{s'}(\mathbf{p}')|^2$. Of course, the contribution of LH helicity of electron to above spin sum will vanish. However, it is still convenient to keep doing the spin sum since it can then be evaluated “easily” in terms of traces of γ -matrices.

Similarly, $R \rightarrow L$ for decay into LH electron.]

(ii) Based simply on Dirac structure of the amplitude, determine the corresponding helicities of the anti-fermion (positron).

[Hint: Pay special attention to the relation between helicity/chirality of the v -spinor and that of the corresponding positron state (see end of section 4.6 of Lahiri and Pal or the corresponding class notes).]

Naively, there seem be *four* possible helicity combinations for the final state, namely, two spin states each for electron and positron. However, what we find here is that two of these

helicity combinations vanish.

(iii) Is the above correlation between chiralities/helicities of the electron and positron “expected” based on the conservation of the component of the total (i.e., spin and orbital) angular momentum *along direction of emission of electron* (and the positron)?

In particular, even though the electron-positron is in p -wave (as discussed in lecture), will there be a non-zero component of orbital angular momentum along the beam direction?

(iv) How is *total* angular momentum (i.e., not just the component along direction of motion) conserved in such a decay in general (i.e., *irrespective* of the specific interaction), given that final state (fermion-antifermion) is in p -wave (i.e., $L = 1$) and initial particle has zero angular momentum (spin)?

In other words, what provides the “missing momentum” in above argument?

[**Hint:** you might find it useful to go back to homework problem 5.4.2 (i)-(ii).]

Is your answer to above question consistent with the correlation of helicities obtained (for this specific interaction) in part (ii) above?

(v) Do the above two decay widths (i.e., *squares* of amplitudes) into specific helicities add up to the decay width obtained by summing over spins *without* helicity projection operators?

Was this result expected (or should the two helicity amplitudes non-trivially *interfere* instead)?

[**Hint:** Again, electron is massless here so that chirality is same as helicity. Moreover, helicity eigenstates are energy eigenstates (actually, this is true even for the case of massive electron). So, think about whether decays to different helicities constitute *different* final states (or not).]

8.2.2 Modified interaction

Do *not* neglect mass of electron in this problem.

Instead of the Yukawa interaction studied in lecture, i.e., Eq. 6.1 of Lahiri and Pal, consider the Lagrangian:

$$\mathcal{L}_{int} = h\phi\bar{\psi}\gamma_5\psi, \tag{37}$$

where ϕ is still a *real* scalar.

Note that coupling constant has to be imaginary in order for this to be hermitian [as discussed in homework problem 6.1.1 (ii)].

(i) Is the above interaction parity invariant if we choose intrinsic parity of ϕ to be *odd*?

[**Hint:** Use result of homework problem 6.1.4 (i).]

(ii) Calculate the (total) decay width of scalar particle (in its rest frame) into electron-positron pair using this interaction.

(**Hint:** re-use the intermediate steps – no need to rewrite the derivations of those steps – from the calculation of decay width for the interaction in Eq. 6.1 of Lahiri and Pal which was presented in lecture or in Lahiri and Pal.)

Compare to the case of the interaction in Eq. 6.1 of Lahiri and Pal that was also discussed in lecture, especially which partial wave the electron-positron pair is in.

(iii) Is the electron-positron being in s -wave forbidden for such a decay based on “symmetries”, i.e., parity invariance and/or angular momentum conservation (i.e., *irrespective* of the specific form of the interaction)?

[**Hint:** repeat the analysis of homework problem 5.4.2 (i)-(ii) – where we did not specify the form of the interaction – for this new case.]

(iv) Does your above *explicit* calculation of the decay width in part (ii) above (i.e., a *specific* interaction) confirm the *general* expectation in part (iii) above?

8.2.3 Most general Yukawa-type interaction

Do *not* neglect mass of electron in this problem, except in the very last part.

Combine the above two types of Yukawa interactions to form the most general Yukawa-type Lagrangian:

$$\begin{aligned}\mathcal{L}_{int} &= -\bar{\psi}(h_S + h_P\gamma_5)\psi\phi \\ &= -\bar{\psi}(h_L L + h_R R)\psi\phi\end{aligned}\tag{38}$$

where $L, R = (1 \mp \gamma_5)/2$ as usual, i.e., $h_{S,P} = (h_R \pm h_L)/2$. Here, the subscript “ S ” (P) on h in first (second) term of the above Lagrangian denotes the corresponding fermion-bilinear being a scalar (pseudo-scalar).

[**Hint:** Refer to homework problem 6.1.1 for the complex nature of $h_{S,P}$ (based on hermiticity of the Lagrangian).]

(i) In the *rest* frame of the scalar particle, calculate the total decay width into electron and positron.

(ii) Show that for $h_L = h_R$ (or equivalently, $h_P = 0$), you recover the result derived in lecture, i.e., Eq. 7.34 of Lahiri and Pal and that for $h_L = -h_R$ (or equivalently, $h_S = 0$) you recover instead the result you derived in problem 8.2.2 (ii) above.

(iii) **Interference of two chiralities:** rewrite the total decay width in terms of contributions to the decay width which are proportional to $|h_L|^2$ and $|h_R|^2$ (i.e., corresponding to the two “*chirality* amplitudes” *separately*) and a contribution to the decay width which is proportional to $(h_L h_R^* + h_L^* h_R)$, i.e., which corresponds to an *interference* between the two chirality amplitudes for the electron.

Was this interference expected or not (i.e., should the two chiralities add *incoherently* instead in the decay width)? Again, mass of electron is *not* to be neglected so that chirality \neq helicity.

(iv) What happens to the above interference contribution, i.e., $\propto (h_L h_R^* + h_L^* h_R)$, if we set the electron mass to zero (i.e., chirality *is* same as helicity)?

Was this result expected?

8.3 “Measuring” chirality

As discussed in lecture, in this problem, we will consider the observable effects of various choices the “chiral structure” (see more below) of the 4-fermion interaction. In some sense, these different results for the scattering cross-sections gives “physical” meaning to (certainly allows a measurement of) chirality.

Consider the (general) form of the interaction:

$$\mathcal{L}_{int} = 2\sqrt{2}G_F \overline{\Psi}_{(\nu_e)} \gamma^\lambda P_e \psi_{(e)} \overline{\Psi}_{(\mu)} \gamma_\lambda P_\mu \psi_{(\nu_\mu)}. \quad (39)$$

In the following, **calculate** the *differential* cross-section for the process:

$$e^-(p) + \nu_\mu(k) \rightarrow \mu^-(p') + \nu_e(k') \quad (40)$$

in the center-of-mass frame, i.e., analog of Eq. 7.130 of Lahiri and Pal, for various choices of $P_{e, \mu}$.

Assume that both neutrinos are massless (as in lecture/Lahiri and Pal), but that they do have *both* R and L chiralities (as done in lecture, but *unlike* in real world/Lahiri and Pal).

Note that in lecture, we already discussed the case $P_e = L$, i.e., $(1 - \gamma_5)/2$ and $P_\mu = R$, i.e., $(1 + \gamma_5)/2$. And in Lahiri and Pal, the case $P_e = P_\mu = L$ is worked out for which the final result is given in Eq. 7.130.

Note that chirality is *not* same as helicity in these examples, where fermion masses are not to be neglected compared to the center-of-mass energy [except for the parts 8.3.2 (ii) and 8.3.3 (ii)].

(**Hint:** again, re-use the intermediate steps – no need to rewrite the derivations of those steps – from the calculation of the case $P_e = R$ and $P_\mu = L$ presented in lecture or of the case $P_e = P_\mu = L$.)

8.3.1 R for both electron and muon parts

(i) $P_e = P_\mu = R$

Compare with Eq. 7.130, where $P_e = P_\mu = L$ instead of R as in this problem (there is also a factor of 2 difference due to our assumption of neutrinos having both helicities vs. only one in Lahiri and Pal.)

8.3.2 R for electron part and L for muon part

(i) $P_e = R, P_\mu = L$, i.e., “switched” with respect to lecture.

(ii) *In the limit that the electron and muon masses can be neglected compared to the center-of-mass energy (so that chirality is same as helicity), does this differential cross-section vanish when the muon goes along the same direction as the incoming electron?*

Along the lines of discussion in lecture, show that this result of part (ii) above is expected based on angular momentum conservation applied along the beam direction.

[**Hint:** as usual, there are two contributions to angular momentum along any direction: spin and orbital.]

Also, clearly, electron in initial state is left-handed (LH) and similarly muon neutrino is RH. You will have to figure out chiralities/helicities of the muon and electron neutrino in final state, based on the Dirac structure of the interaction/amplitude.]

8.3.3 No L, R

(i) $P_e = P_\mu = \mathbf{1}$, i.e., $(L + R)$.

(ii) *In the massless limit for both electron and muon, compare the result for this case to the sum of the cross-sections (i.e., $\propto |\text{amplitude}|^2$) for the the four earlier cases with specific chiralities for the two parts, i.e., (a) $P_e = R, P_\mu = L$ discussed in lecture, (b) $P_e = P_\mu = L$ discussed in Lahiri and Pal, (c) $P_e = P_\mu = R$, discussed in problem above and (d) $P_e = L, P_\mu = R$, discussed in problem above.*

Was the result of this comparison as expected [or should the various chirality amplitudes – for cases (a)-(d) above – interfere (see discussion in the example of decay above)]?

9 Homework 9 (Quantization of EM field and gauge invariance), due Wednesday, December 1

9.1 Commutation relations for creation/annihilation operators vs. those for fields

Exercise 8.4 of Lahiri and Pal.

9.2 Hamiltonian and momentum in terms of creation/annihilation operators

(The calculations below are analogous to ones done earlier for scalar field.)

(i) Calculate the Hamiltonian of the electromagnetic field (in 't Hooft-Feynman gauge)

$$\begin{aligned}
 H &= \int d^3x (\Pi_{A_\mu} \partial_0 A_\mu - \mathcal{L}) \text{ with} \\
 \mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2
 \end{aligned} \tag{41}$$

in terms of the creation/annihilation operators for photons.

This expression for the Hamiltonian was used in lecture.

(ii) Similarly, calculate the momentum of the electromagnetic field

$$\begin{aligned}
 P_i &= \int d^3x T^{0i} \text{ with} \\
 T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^\rho)} \partial^\nu A^\rho - g^{\mu\nu} \mathcal{L}
 \end{aligned} \tag{42}$$

in terms of these operators.

(iii) Show that (just like for Hamiltonian) only the *transversely* polarized photon states contribute to the expectation value/matrix element of momentum operator between any two *physical* states.

9.3 Residual gauge invariance in Lorenz gauge

Suppose $|\Psi_T\rangle$ is a state which contains only transverse photons.

(i) Show that (easy!) this state satisfies the Gupta-Bleuler condition, i.e.,

$$(a_3(\mathbf{k}) - \mathbf{a}_0(\mathbf{k})) |\Psi_T\rangle = 0 \tag{43}$$

Define a new state by

$$|\Psi'_T\rangle = \left(1 + c [a_3^\dagger(\mathbf{k}) - \mathbf{a}_0^\dagger(\mathbf{k})]\right) |\Psi_T\rangle \tag{44}$$

where c is a constant.

(ii) Show that this new state also satisfies the Gupta-Bleuler condition.

(iii) Show that replacing $|\Psi_T\rangle$ by $|\Psi'_T\rangle$ corresponds to a gauge transformation (which keeps A^μ *within* Lorenz gauge), i.e.,

$$\langle \Psi'_T | A^\mu(x) | \Psi'_T \rangle = \langle \Psi_T | A^\mu(x) + \partial^\mu \theta(x) | \Psi_T \rangle \tag{45}$$

where $\theta(x) \sim \text{Re}(ic e^{-ikx})$, i.e., up to normalization factors.

This shows (as mentioned in lecture) that one combination of the (“unphysical”) longitudinal and scalar polarizations/degrees of freedom – the one which is *not* removed by the Lorenz/Gupta-Bleuler condition – corresponds to arbitrariness in the choice of Lorenz gauge, i.e., the remnant freedom to perform gauge transformations where $\theta(x)$, the parameter of the gauge transformation, satisfies $\partial_\mu \partial^\mu \theta(x) = 0$.

9.4 Photon propagator

Determine the Feynman propagator for photon in momentum space, i.e., show that

$$\langle 0|T[A^\mu(x)A^\nu(x')]|0\rangle = -\frac{g^{\mu\nu}}{(2\pi)^4} \int d^4k \frac{e^{-ik \cdot (x-x')}}{k^2 + i\epsilon} \quad (46)$$

by plugging in the expansion for A_μ in terms of creation/annihilation operators and then using the commutation relations for the latter.

(**Hint:** be careful with the ζ factors in the commutation relations for these operators and in the orthonormality/completeness relations for polarization vectors.)

9.5 Gauge invariance for charged scalar field

Exercise 9.1 of Lahiri and Pal.

9.6 Covariant derivative for fermion field

Exercise 9.2 of Lahiri and Pal.

9.7 Axial-vector current

We can consider another current (as follows) which is bilinear in fermion fields, but the question is whether it can be consistently coupled to electromagnetic field.

Consider the transformation (compare to the “usual” one, for example, in Eq. 9.4 of Lahiri and Pal) given by

$$\psi(x) \rightarrow e^{i\alpha\gamma^5} \psi(x) \quad (47)$$

(where α is a constant) with the Lagrangian given by

$$\mathcal{L} = \bar{\psi} (i\partial_\mu \gamma^\mu - m) \psi \quad (48)$$

(i) Calculate the Noether current corresponding to this transformation, i.e., defined (in general) as

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \quad (49)$$

where $\Delta \phi$ is the shift in the field.

(ii) Which (if any) of the two terms of the Lagrangian (the one with a derivative – also called “kinetic” – or the one with mass) is invariant under this transformation?

(iii) If the *total* Lagrangian is *not* invariant (in general) under this transformation, then is there a “limit” where it *is* invariant?

- (iv) Calculate the divergence of the current obtained in part (i) above.
- (v) Does your result in part (iv) agree with the expectations from parts (ii) and (iii) above? In particular, if the current is not conserved, i.e., if its divergence is non-zero (in general), then is there a limit where it is conserved?
- (vi) Find the *separate* transformation of the two chiralities of ψ under the transformation of the full ψ .
- (vii) Show that the mass term in the Lagrangian corresponds to a “coupling” of the two different chiralities.
- (viii) Is the result of parts (ii) and (iii) above then consistent with the expectation based on parts (vi) and (vii) above?
- (ix) Finally, can the above current be coupled to the electromagnetic field?

10 Homework 10 [Specific calculations of S -matrix elements (II)], due Wednesday, December 8

10.1 Helicity/chirality amplitudes for $e^+e^- \rightarrow \mu^+\mu^-$

Use the center-of-mass frame and neglect both electron and muon masses (compared to the center-of-mass energy) so that helicity is same as chirality for this problem.

(i) As discussed in class, calculate the *differential* cross-section for the four combinations of chiralities of incoming electron and outgoing muon, namely

- (a) $e_L^- + e^+ \rightarrow \mu_L^- + \mu^+$
- (b) $e_L^- + e^+ \rightarrow \mu_R^- + \mu^+$
- (c) $e_R^- + e^+ \rightarrow \mu_L^- + \mu^+$
- (d) $e_R^- + e^+ \rightarrow \mu_R^- + \mu^+$.

[**Hint:** Just like in problem 8.2.1 (i), simply insert L, R projection operators appropriately in the spin sum which appeared in the calculation of *unpolarized* cross-section.]

(ii) For each case, determine the chirality of the positron and anti-muon (using the Dirac structure of the amplitude), paying special attention [just as in problem 8.2.1 (ii)] to the relation between helicity/chirality of the v -spinor and that of the corresponding positron state (see end of section 4.6 of Lahiri and Pal or the corresponding class notes).

(iii) In each case, you should find that the differential cross-section vanishes either in the forward or backward direction. Is this result expected based on the conservation of the component of the *total* (i.e., spin and orbital) angular momentum *along beam direction*?

(iv) Does the sum of the above four cross-sections (i.e., *squares* of the amplitudes) equal the unpolarized differential cross-section that was calculated in class? Again, was this expected (or should the chirality amplitudes *interfere* instead)?

10.2 Electron-muon elastic scattering

Neglecting the mass of *electron* (but *not* of muon), calculate the *differential* cross-section for the process

$$e^- + \mu^- \rightarrow e^- + \mu^- \quad (50)$$

in the center-of-mass frame, with *unpolarized* electrons and muons.

(You should find that the amplitude for this process is related to that for $e^+e^- \rightarrow \mu^+\mu^-$ by “crossing” symmetry, i.e., the amplitude for a process involving a particle with momentum p in the initial state is equal to that for an otherwise identical process but with an *antiparticle* of momentum $-p$ in final state.)

10.3 A *number* (!) for cross-section

10.3.1 Basic/benchmark cross-section

(All cross-sections below are total, unpolarized and in center-of-mass frame. Neglect masses of fermions relative to the center-of-mass energy in these problems.)

Calculate the cross-section for $e^+e^- \rightarrow \mu^+\mu^-$ at the center-of-mass energy of 100 GeV (typical energy of the collider at CERN in Geneva, Switzerland which was in operation in 1990’s).

(Use $\alpha \approx 1/135$.)

Convert the cross-section from natural units (i.e., $c = 1$ and $\hbar/(2\pi) = 1$) to (usual) units, i.e., an *area*.

10.3.2 Scaling of cross-section with charge and energy

(i) At what energy will the cross-section be twice that in above part?

(ii) Suppose we annihilate a down “quark” (an elementary particle) of charge $-1/3$ (and its anti-quark) to produce an up quark with charge $2/3$ (and its anti-quark). At what center-of-mass energy will the cross-section be same as in part (i) above?

(For those of who have heard about it, *ignore* the “color” degrees of freedom of quarks and anti-quarks in above calculation.)

(iii) Suppose the value of the *fundamental* charge e was *twice* as large as in the real world. What will be the new cross-section for the process in part (ii) above (with the other parameters remaining the same)?