PHY 624 (Fall 2010): Advanced Quantum Mechanics

Take-home Final

General guidelines:

(i) The final is **due by noon, Wedesday, December 15** in the box outside my office, Rm. 4119.

(ii) The four problems (with a few parts in each) are (roughly) in increasing order of difficulty/length of calculation.

(iii) All concepts, identities etc. you need for solving these problems are contained in lecture/homework notes and in Lahiri and Pal.

(iv) You can simply "borrow" (i.e., no need to repeat the derivation) any results for cross-sections or Feynman amplitudes or phase space factors from lecture/homework/Lahiri and Pal for this purpose.

(v) For these scattering problems, work to second order in coupling constant, say e (in the Feynman amplitude).

(vi) You should **avoid** discussing with other students specifically about these problems.

(vii) Before you ask questions specifically about these problems, please read the statements of the problems (and especially hints) very carefully. Again, there are no trick questions and no new concepts involved here. So, many of your questions might be answered if you think more about them!

(viii) If you still have questions or clarifications, I prefer that you try to ask them during lecture (for *quick* ones) or (for more detailed ones) during special office hours on Thursday, December 9 and Monday, December 13 (both 2-3 pm. and in Rm. 4102) – that way all students can be informed about these issues.

(ix) If you are unable to have all your questions answered during these times, then you can send me email or come to my office, Rm. 4119 (it will be better if you send me an email first to set up a time).

(x) Please write clearly and show all steps in calculations.

(xi) Please try to use the notation (for angles, masses, momenta of particles etc.) – even if it is for intermediate steps – which is specified in the statement of the problems.

(xii) You are allowed to use mathematica (or look up tables) for doing integrals, although you should be able to do it "by hand".

1 Scalar-scalar scattering in QED

Consider scalar QED, i.e.,

$$\mathcal{L} = (D^\mu \phi)^\dagger (D_\mu \phi) - m_\phi^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where

$$D_\mu = \partial_\mu - ieA_\mu \tag{2}$$

and ϕ is a complex scalar field (with electric charge -1).

Determine the Feynman *amplitude* (no need to calculate the cross-section) for the process:

$$\phi^- (p_1) + \phi^- (p_2) \rightarrow \phi^- (p_1') + \phi^- (p_2') \tag{3}$$

where $\phi^\mp$ here denote spin-0 particle of electric charge ∓ 1.

Simply use the Feynman Rules for scalar QED given in notes.

Use $\xi = 1$, i.e., 't Hooft-Feynman gauge for photon propagator.

Write your answer in terms of various p's, coupling constants etc. (*no* need to go to center-of-mass frame etc.)

(**Hint**: is there only one Feynman diagram? If not, then is there a relative sign between the Feynman amplitudes for the various diagrams? The result of homework problem 7.4 might be useful for this purpose.)

2 Forward-Backward Asymmetry

In the center-of-mass reference frame, the forward-backward asymmetry (denoted by A_{FB}) for the process

$$e^+ + e^- \rightarrow \mu^+ + \mu^- \tag{4}$$

(and in general, for any $2 \rightarrow 2$ process) is defined as:

$$A_{FB} = \frac{\sigma (0 < \theta < \pi/2) - \sigma (\pi/2 < \theta < \pi)}{\sigma (0 < \theta < \pi/2) + \sigma (\pi/2 < \theta < \pi)}, \tag{5}$$

where θ is angle between *incoming* e^- *and outgoing* μ^- *directions* so that $\sigma (0 < \theta < \pi/2) \equiv \int_{\theta=0}^{\theta=\pi/2} d\sigma$ is the cross-section for production of a muon in "forward" direction and so on.

(i) Neglecting $m_{e,\mu}$ compared to the center-of-mass energy, calculate A_{FB} in QED.

(ii) Neglecting $m_{e,\mu}$ compared to the center-of-mass energy, calculate A_{FB} for a theory with the following interaction *only* (which is a generalization of QED to the case of a new massive vector boson), i.e., do *not* add QED interaction here:

$$\begin{aligned} H_{int} &= \int d^3x \left[\overline{\psi_{(e)}} \gamma^\mu \left(g_V^{(e)} + g_A^{(e)} \gamma^5 \right) \psi_{(e)} + \overline{\psi_{(\mu)}} \gamma^\mu \left(g_V^{(\mu)} + g_A^{(\mu)} \gamma^5 \right) \psi_{(\mu)} \right] Z_\mu \\ &= \int d^3x \left[\overline{\psi_{(e)}} \gamma^\mu \left(g_L^{(e)} L + g_R^{(e)} R \right) \psi_{(e)} + \overline{\psi_{(\mu)}} \gamma^\mu \left(g_L^{(\mu)} L + g_R^{(\mu)} R \right) \psi_{(\mu)} \right] Z_\mu \end{aligned} \tag{6}$$

where the field whose quanta are the massive gauge bosons is denoted by Z_μ and L, $R = (1 \mp \gamma_5)/2$ as usual so that g_V, g_A denote couplings to vector and axial-vector fermionic currents, respectively. Equivalently, g_L, g_R denote couplings to left and right-handed fermionic currents, respectively, i.e., $g_{V,A} = (g_R \pm g_L)/2$. Note that these couplings can in general be different for e^- and μ^-, i.e., $g_V^{(e)} \neq g_V^{(\mu)}$ for example.

All couplings here are **real** and the interaction does *not* couple an electron to a muon (just like in QED).

You should be able to "guess" most of the Feymnan rules for this theory by simply generalizing the ones for QED, with the exception of the Z boson propagator which can be *assumed* to be $-ig_{\mu\nu}/\left(q^2 - m_Z^2 + i\epsilon\right)$ (where q is the momentum flowing through the Z boson line).

Hint: Based on your above calculation of A_{FB} for QED, it should be clear that the angular dependence of Feynman amplitude/cross-section is of most relevance for A_{FB} (i.e., factors such as π's from phase space are not important since A_{FB} is a *ratio* of cross-sections.)

Hint: Instead of using "brute force" for this case, try to break-up the total cross-section into helicity (= chirality, since $m_{e,\,\mu}$ is being neglected here) amplitudes, just like you did for QED in homework problem 10.1. *You should be able to simply generalize the helicity/chirality amplitudes from QED for this case.*

(iii) If you set $g_V^{e,\,\mu} = e$ and $g_A^{e,\,\mu} = 0$, then does your result reduce to what you calculated in (i) above for QED?

(Such asymmetries provide a crucial test of the Standard Model.)

3 e^+e^- annihilation into $\mu^+\mu^-$ in Yukawa theory

Consider *both* electrons and muons coupled to a (real) scalar (Yukawa theory):

$$\begin{aligned} \mathcal{L} &= \overline{\psi_{(e)}}\left(i\gamma^\mu\partial_\mu - m_e\right)\psi_{(e)} + \overline{\psi_{(\mu)}}\left(i\gamma^\mu\partial_\mu - m_\mu\right)\psi_{(\mu)} + \\ &\quad \frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi\right) - \frac{1}{2}m_\phi^2\phi^2 + \\ &\quad \phi\left(h_e\overline{\psi_{(e)}}\psi_{(e)} + h_\mu\overline{\psi_{(\mu)}}\psi_{(\mu)}\right) \end{aligned} \tag{7}$$

(i) In the center-of-mass frame, calculate the *total* cross-section (*un*polarized) for the process

$$e^+ + e^- \;\rightarrow\; \mu^+ + \mu^- \tag{8}$$

Neglect the electron mass, but *not* the muon mass relative to the (total) center-of-mass energy, $\sqrt{s}$.

Write the final result in terms of $\sqrt{s}$, m_μ and m_ϕ.

(**Hint**: you might be able to re-use – with caution! – some of the steps from the calculation of scalar decay into e^+e^-.)

(ii) Is the $\mu^+\mu^-$ pair in s or p-wave?

Use the following diagnostic for orbital angular momentum of the $\mu^+\mu^-$ (i.e., *final*) state (similar to the case of decay of scalar into fermion-antifermion): cross-section (σ) scales as β^1 for s-wave and β^3 for p-wave, where β is the speed of the μ^+ (and μ^-).

(iii) Was the above result about orbital angular momentum of $\mu^+\mu^-$ expected (based on parity invariance and angular momentum)?

[**Hint**: can the expectation of fermion-antifermion being in p-wave for (on-shell) scalar decay of problem 5.4.2 (i) and (ii) be used here (even though the scalar here is *off*-shell)?]

4 e^+e^- annihilation into scalars in QED

Consider the combination of "usual" and scalar QED, i.e.,

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m_e\right)\psi + \left(D^\mu\phi\right)^\dagger\left(D_\mu\phi\right) - m_\phi^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9}$$

where D_μ is as in Eq. (2).

(i) In the center-of-mass frame, calculate the *differential* cross-section (*un*polarized) for the process

$$e^+ + e^- \rightarrow \phi^- + \phi^+ \tag{10}$$

Just like in problem 1, ϕ in Eq. (9) denotes a complex scalar field, whereas $\phi^\mp$ in Eq. (10) denotes spin-0 particles of electric charge ∓ 1.

Neglect the electron mass, but *not* the scalar mass relative to the total center-of-mass energy, $\sqrt{s}$.

Write the final result in terms of $\sqrt{s}$, θ (angle between the ϕ^- and e^-) and m_ϕ.

Just like in problem 1, simply use the Feynman Rules for scalar QED given in notes and use $\xi = 1$, i.e., 't Hooft-Feynman gauge for photon propagator.

(**Hint**: be careful with signs of 4-momenta appearing in the scalar-photon vertex factor. As shown in notes on scalar QED, it is the *sum* of 4-momenta in the two scalar lines, but both being *along the direction of negative charge flow*, which is assigned to this vertex.)

(ii) Does the cross-section vanish when the ϕ^- is emitted *either* in the forward or backward direction (i.e., $\theta = 0$ or π)?

(iii) Was this result expected based on angular momentum conservation (for the component along the direction of the incoming particles)?

[**Hint**: it might be useful to "break down" the cross-section in terms of the helicity (= chirality) amplitudes, as was done in homework problem 10.1.]

(iv) Can we add a Yukawa-type interaction, i.e., between the electron and the scalar, in this theory?