Solutions for Take-home Exam for Physics 623, Spring 2010, O.W. Greenberg

1. (60 points) A system has two degenerate states $|l_1\rangle, |l_2\rangle$ of energy 0 and a state $|l_3\rangle$ of high energy M. The degenerate states are coupled to each other with coupling a and to the high energy state with couplings b_1 and b_2, labeled by the corresponding zero energy state.
(a) Write the Hamiltonian for this system.

$$H = M||l_3\rangle\langle l_3| + a(|l_1\rangle\langle l_2| + |l_2\rangle\langle l_1|) + b_1(|l_1\rangle\langle l_3| + |l_3\rangle\langle l_1|) + b_2(|l_2\rangle\langle l_3| + |l_3\rangle\langle l_2|) \quad (1)$$

or $H = H_0 + V$,

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M \end{pmatrix} \quad (2)$$

$$V = \begin{pmatrix} 0 & a & b_1 \\ a & 0 & b_2 \\ b_1 & b_2 & 0 \end{pmatrix} \quad (3)$$

in an obvious notation. Can choose a, $b_{1,2}$ real by absorbing phases in the definitions of the states.
(b) Use degenerate perturbation theory to find the first-order eigenvalues for this system.
We must diagonalize the Hamiltonian in the degenerate states. The secular equation in the degenerate subspace is

$$\begin{vmatrix} -E & a \\ a & -E \end{vmatrix} = 0 \quad (4)$$

The solutions are, for first order eigenvalue a,

$$|l_+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (5)$$

and for the first order eigenvalue $-a$,

$$|l_-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{6}$$

. The first-order eigenvalues are

$$\langle l_+|V|l_+\rangle = a, \quad \langle l_-|V|l_-\rangle = -a \tag{7}$$

Use the states $|l_+\rangle$, $|l_-\rangle$, $|l_3\rangle$ as new zeroth basis states.

(c) Transform the Hamiltonian to the new basis,

$$H = \begin{pmatrix} a & 0 & (b_1+b_2)/\sqrt{2} \\ 0 & -a & (b_1-b_2)/\sqrt{2} \\ (b_1+b_2)/\sqrt{2} & (b_1-b_2)/\sqrt{2} & M \end{pmatrix} \tag{8}$$

(c) Use degenerate perturbation theory to find the first-order eigenstates for this system. Hint: the zero energy eigenstates get corrections both in the high-energy sector and in the degenerate sector.
Rename the basis states above as zeroth-order states, $|l_+^{(0)}\rangle$, $|l_-^{(0)}\rangle$, $|l_3^{(0)}\rangle$.

$$|l_+^{(1)}\rangle = \frac{b_1^2 - b_2^2}{4aM}|l_-^{(0)}\rangle - \frac{b_1+b_2}{\sqrt{2}M}|l_3^{(0)}\rangle, \tag{9}$$

$$|l_-^{(1)}\rangle = -\frac{b_1^2 - b_2^2}{4aM}|l_+^{(0)}\rangle - \frac{b_1-b_2}{\sqrt{2}M}|l_3^{(0)}\rangle, \tag{10}$$

$$|l_3^{(1)}\rangle = \frac{b_1+b_2}{\sqrt{2}M}|l_+^{(0)}\rangle + \frac{b_1-b_2}{\sqrt{2}M}|l_-^{(0)}\rangle. \tag{11}$$

(d) Use degenerate perturbation theory to find the second-order eigenvalues for this system.
In general, $\Delta^{(2)} = \langle l^{(0)}|V|l^{(1)}\rangle$.

$$\Delta_+^{(2)} = -\frac{(b_1+b_2)^2}{2M}, \tag{12}$$

$$\Delta_+^{(2)} = \frac{(b_1 - b_2)^2}{2M}, \tag{13}$$

$$\Delta_3^{(2)} = -\frac{b_1^2 + b_2^2}{M}. \tag{14}$$

(e) For the special choice $b_1 = b_2 = b$ find a relation between a and b that makes the eigenvalue equation quadratic (and thus easily solvable).
With $b_1 = b_2 = b$ the secular equation can be solved, but it is easier to solve with the special choice $a = 2b^2/M$. For that case, the eigenvalues are $-a,\ 0,\ M + a$.

(f) Solve the case of (e) exactly and compare with your results in (b) and (c).
The solutions with the simplified Hamiltonian are, for $E = -a$,

$$|l_-\rangle = |l_-^{(0)}\rangle, \tag{15}$$

for $E = 0$,

$$|l_+\rangle = |l_+^{(0)}\rangle - \frac{\sqrt{2}b}{M}|l_3^{(0)}\rangle, \tag{16}$$

for $E = M + a$,

$$|l_3\rangle = |l_3^{(0)}\rangle + \frac{\sqrt{2}b}{M}|l_+^{(0)}\rangle. \tag{17}$$

These agree with the results of (b) and (c).
2. (20 points) A spin 1/2 particle with magnetic moment μ is in a constant magnetic field $\mathbf{B_0}$. A time-dependent magnetic field $\mathbf{B_1}exp(-i\omega t)$ is turned on for a time T. The field $\mathbf{B_1}$ is in the plane orthogonal to $\mathbf{B_0}$. If the magnetic moment is in its ground state initially, estimate the spin-flip probability at the end of time T.
See handwritten page.

3. (20 points) Use first-order time-dependent perturbation theory to calculate the probability that an harmonic oscillator in its ground state in the infinite past ends
(a) in its first excited state in the infinite future if it is excited by a time-dependent *force* (not potential)

$$F(t) = \frac{F_0\tau/\omega}{\tau^2 + t^2}. \tag{18}$$

(b) in its second excited state over the same time interval.

(c) What is the lowest order of time-dependent perturbation theory for which the oscillator will end in its nth excited state?

(a) We take the potential to be $-xF(t)$. First-order perturbation theory gives

$$c_n^{(1)}(t) = (-i/\hbar)\int_{t_0}^{t} \exp(i\omega_{ni}t')V_{ni}(t')dt'. \tag{19}$$

For our case, $t_0 = -\infty$, $t = \infty$, $\omega_{ni} = (E_n - E_i)/\hbar$, and $E_i = E_0 = \hbar\omega/2$, $E_n = E_1 = 3\hbar\omega/2$, so

$$c_i^{(1)}(\infty) = \frac{iF\tau}{\hbar\omega}\langle 1|x|0\rangle \int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{\tau^2 + t^2}dt \tag{20}$$

To evaluate the integral, we note that if $t \to t + i\epsilon$ the exponential factor is damped for $\epsilon > 0$, so we complete the contour in the upper half plane and use the residue at $\tau = it$ to find

$$\int_{-\infty}^{\infty} \frac{exp(i\omega t)}{\tau^2 + t^2}dt = \frac{2\pi i}{2i\tau}exp(-\omega\tau). \tag{21}$$

The matrix element is $\langle 1|x|0\rangle = \sqrt{\hbar/2m\omega}$. The amplitude becomes

$$c_i^{(1)}(\infty) = \frac{i\pi F}{\sqrt{2m\hbar\omega^3}}e^{-\omega\tau}. \tag{22}$$

The probability is

$$|c_i^{(1)}(\infty)|^2 = \frac{(\pi F)^2}{2m\hbar\omega^3}e^{-2\omega\tau}. \tag{23}$$

(b) The operator x changes the state of the oscillator by 1, so the second excited state is not reached in 1st order.

(c) The nth excited state is first excited in nth order.