5.1. (a) The first order correction is via (5.1.37) just $\langle 0|bx|0\rangle = 0$. The second order correction for the energy is (c.f. (5.1.42) and (5.1.43))

$$\Delta E = -\sum_n \frac{|\langle n|bx|0\rangle|^2}{E_n - E_o} = -b^2 \sum_n \frac{|\langle n|x|0\rangle|^2}{E_n - E_o},$$

where $E_n = (n+\frac{1}{2})\hbar\omega$. Now $\langle n|x|0\rangle = \sqrt{\hbar/2m\omega}\,\delta_{n1}$, so $\Delta E = -b^2(\sqrt{\hbar/2m\omega})^2/(E_1 - E_o) = -b^2/2m\omega^2$ is the energy shift, and the energy of the ground state becomes $E^{(o)} = \frac{1}{2}\hbar\omega + \Delta E = \frac{1}{2}\hbar\omega - b^2/2m\omega^2$.

(b) The Schrödinger equation for this problem is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (\tfrac{1}{2}m\omega^2 x^2 + bx)\psi = E^{(o)}\psi.$$

Let $x' = x + b/m\omega^2$, than above equation can be reduced to

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx'^2} + \tfrac{1}{2}m\omega^2[x'^2 - (b/m\omega^2)^2]\psi = E^{(o)}\psi$$

that is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx'^2} + \tfrac{1}{2}m\omega^2 x'^2\psi = (E^{(o)} + b^2/2m\omega^2)\psi.$$

This is again a SHO equation with $E' = E^{(o)} + b^2/2m\omega^2$. For lowest energy value $E' = \frac{1}{2}\hbar\omega$, hence $E^{(o)} = \frac{1}{2}\hbar\omega - b^2/2m\omega^2$ which is exactly the same as the perturbation result in (a).

5.2. From (5.1.44) with $k \leftrightarrow n$ and $\lambda \to g$, we have

$$|k\rangle = |k^{(o)}\rangle + g\sum_{n\neq k}\frac{|n^{(o)}\rangle V_{nk}}{E_k^{(o)} - E_n^{(o)}} + \cdots\cdots\cdots$$

Using orthonormality of $|k^{(o)}\rangle$ and $|n^{(o)}\rangle$ we have

$$\langle k|k\rangle = 1 + g^2\sum_{n\neq k}\frac{|V_{nk}|^2}{(E_k^{(o)} - E_n^{(o)})^2} + \cdots\cdots\cdots$$

and

$$\frac{|\langle k|k^{(o)}\rangle|^2}{|\langle k|k\rangle|^2} = 1 - g^2\sum_{n\neq k}\frac{|V_{nk}|^2}{(E_k^{(o)} - E_n^{(o)})^2} + O(g^3)$$

Prob. that perturbed eigenstate is in the unperturbed eigenstate = prob that the unperturb. eigenstate is in the perturbed eigenstate.

5. 3. Solving the Schrödinger equation for the unperturbed system, we can easily find the energy eigenfunctions. They are $\psi_G = \sqrt{2/L}\sqrt{2/L}\ \sin\pi x/L\ \sin\pi y/L = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}$ for ground state, and $\psi_{e1}^{(1)} = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L}$ or $\psi_{e1}^{(2)} = \frac{2}{L}\sin\frac{2\pi x}{L}\ \sin\frac{\pi y}{L}$ for the first excited state. So obviously the zeroth order eigenfunction for the ground state is just $\psi_G = \frac{2}{L}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}$, with the first order energy shift of $<1|\lambda xy|1> = \int_0^L\int_0^L \frac{4}{L^2}\lambda xy\sin^2\pi x/L\ \sin^2\pi y/L\ dxdy = \frac{1}{4}\lambda L^2$, i.e. $\Delta E^{(o)} = \lambda L^2/4$. For the first excited state, there is degeneracy and the perturbation in general lifts the degeneracy. We need to construct the perturbation matrix by evaluating

$$<\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(1)}> = \frac{4\lambda}{L^2}\int_0^L\int_0^L xy\sin^2\pi x/L\ \sin^2 2\pi y/L\ dxdy = \tfrac{1}{4}\lambda L^2$$

$$<\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(2)}> = \frac{4\lambda}{L^2}\int_0^L\int_0^L xy\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\sin\frac{2\pi y}{L}\sin\frac{\pi y}{L}dxdy = \frac{4^4}{81}\lambda L^2/\pi^4$$

while by symmetry $<\psi_{e1}^{(2)}|V_1|\psi_{e1}^{(2)}> = <\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(1)}>$ and $<\psi_{e1}^{(2)}|V_1|\psi_{e1}^{(1)}> = <\psi_{e1}^{(1)}|V_1|\psi_{e1}^{(2)}>$. So the perturbation matrix is

$$\Delta = \frac{\lambda L^2}{4\pi^4}\begin{pmatrix}\pi^4 & 4^5/81\\ 4^5/81 & \pi^4\end{pmatrix}.$$

Diagonalizing Δ with $\det(\Delta-\lambda' I) = 0$ and

$$(\Delta-\lambda' I)\begin{pmatrix}a\psi_{e1}^{(1)}\\ b\psi_{e1}^{(2)}\end{pmatrix} = 0$$

where $a^2 + b^2 = 1$ (normalization), we get $a = 1/\sqrt{2}$, $b = \pm 1/\sqrt{2}$ and

$$\Delta' = \frac{\lambda L^2}{4\pi^4}\begin{pmatrix}\pi^4+4^5/81 & 0\\ 0 & \pi^4-4^5/81\end{pmatrix}.$$

Hence energy shifts for the first excited state are

$$\frac{(\pi^4+4^5/81)\lambda L^2}{4\pi^4} \approx 0.28\lambda L^2 \text{ and } \frac{(\pi^4-4^5/81)\lambda L^2}{4\pi^4} \approx 0.22\lambda L^2, \text{ or } \frac{1}{4} \pm \frac{256}{81\pi^4}$$

with corresponding zeroth order energy eigenfunctions

$$\frac{1}{\sqrt{2}}\frac{2}{L}[\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L} + \sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}] \text{ and } \frac{1}{\sqrt{2}}\frac{2}{L}[\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L} - \sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}]$$

respectively, or $\frac{1}{\sqrt{2}}(\psi_{e1}^{(1)} \pm \psi_{e1}^{(2)})$.

5.4. (a) State vector for energy eigenstate is characterized by $|n_x,n_y\rangle$, and wave function is given by $\psi_{n_x}(x)\psi_{n_y}(y)$ where $\psi_{n_x}(x)$ and $\psi_{n_y}(y)$ are individually wave functions for one dimensional SHO. The energy for the isotropic two dimensional oscillator is just the sum of the energies for one dimensional oscillators, i.e. $E_{n_xn_y} = \hbar\omega(n_x + \frac{1}{2} + n_y + \frac{1}{2})$. The three lowest-lying states are $(n_x,n_y) = (0,0)$, $(1,0)$, $(0,1)$ with energies $\hbar\omega$, $2\hbar\omega$, $2\hbar\omega$, respectively. Evidently the first excited states are doubly degenerate.

(b) The first order energy shift is clearly zero for the ground state (0,0), since $\langle 0,0|xy|0,0\rangle = 0$ because in $\langle 0|x|0\rangle$ (and $\langle 0|y|0\rangle$) $n_x(n_y)$ must change by one unit. For the first excited states we use the formalism of degenerate perturbation theory by diagonalizing $V = \delta m\omega^2 xy$. In the (1,0) and (0,1) basis

$$V = \delta m\omega^2\begin{pmatrix} 0 & x_{10}y_{01} \\ x_{01}y_{10} & 0 \end{pmatrix} = \tfrac{1}{2}\delta\hbar\omega\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and hence behaves like σ_x. By same method as problem 3 above, we get zeroth order energy eigenkets $\frac{1}{2^{1/2}}(|10\rangle+|01\rangle)$ with $\Delta^{(1)} = \frac{1}{2}\delta\hbar\omega$ and $\frac{1}{2^{1/2}}(|10\rangle-|01\rangle)$ with $\Delta^{(1)} = -\frac{1}{2}\delta\hbar\omega$. So to summarize we have ground state $|0,0\rangle$ with energy $E = \hbar\omega$ (no first order shift) and first excited states $\frac{1}{2^{1/2}}(|10\rangle+|01\rangle)$ with $E = (2+\delta/2)\hbar\omega$ and $\frac{1}{2^{1/2}}(|10\rangle-|01\rangle)$ with $E = (2-\delta/2)\hbar\omega$.

(c) Now $m\omega^2(x^2+y^2)/2 + \delta m\omega^2 xy = \frac{m\omega^2}{2}[(1+\delta)(x+y)^2/2 + (1-\delta)(x-y)^2/2]$. Let us rotate coordinates by 45°, than $X \equiv (x+y)/\sqrt{2}$, $Y \equiv (x-y)/\sqrt{2}$. So

$$H = p_X^2/2m + p_Y^2/2m + m[\omega^2(1+\delta)]X^2/2 + m[\omega^2(1-\delta)]Y^2/2$$

and is effectively again a two dimensional SHO with ω replaced by $\sqrt{1\pm\delta}\,\omega$ in the (X,Y) system. The exact energy for the ground state is $\frac{1}{2}\hbar\omega\sqrt{1+\delta} + \frac{1}{2}\hbar\omega\sqrt{1-\delta} = \hbar\omega + O(\delta^2)$. There is therefore no change in energy if only terms linear in δ are kept. The exact energy for $(n_x,n_y) = (1,0)$ is $\hbar\omega\sqrt{1+\delta}(1+\frac{1}{2}) + \hbar\omega\sqrt{1-\delta}\,\frac{1}{2} = \hbar\omega(2+\delta/2) + O(\delta^2)$; similarly for $(n_x,n_y) = (0,1)$, by letting $\delta \to -\delta$, we have exact energy $\hbar\omega(2-\delta/2) + O(\delta^2)$. Ignoring $O(\delta^2)$ contributions, the results are the same as in (b).

5.5 The Hamiltonian for the system is $H = H_o + \frac{1}{2}\epsilon m\omega^2x^2 = p_x^2/2m + \frac{1}{2}(1+\epsilon)m\omega^2x^2$, hence $V_{ko} = \langle k|V|0\rangle = \langle k|\frac{1}{2}\epsilon m\omega^2x^2|0\rangle \propto \langle k|x^2|0\rangle$. So our task is to evaluate $\langle k|x^2|0\rangle$ or x^2_{ko}. Since from (2.3.24) $x = \sqrt{\hbar/2m\omega}(a + a^\dagger)$ where a and $a^\dagger$ satisfy $a|n\rangle = c_-|n-1\rangle$ and $a^\dagger|n\rangle = c_+|n+1\rangle$, then $x|0\rangle = \sqrt{\hbar/2m\omega}(a|0\rangle + a^\dagger|0\rangle) = \sqrt{\hbar/2m\omega}|1\rangle$ while $x^2|0\rangle = (\sqrt{\hbar/2m\omega})^2(a + a^\dagger)|1\rangle = c_1|0\rangle + c_2|2\rangle$. So $V_{ko} \propto \langle k|x^2|0\rangle = c_1\delta_{ko} + c_2\delta_{k2}$, and only V_{oo} and V_{2o} are relevant to our discussion. Explicit evaluation of c_1 and c_2 (remembering that $(a^\dagger/\sqrt{2})|1\rangle = |2\rangle$ from (2.3.21)), we have $c_1 = \hbar/2m\omega$, $c_2 = \hbar\sqrt{2}/2m\omega$. Thus $V_{oo} = \frac{1}{2}\epsilon m\omega^2\langle 0|x^2|0\rangle = c_1\epsilon m\omega^2/2 = \frac{\hbar}{2m\omega}\frac{\epsilon m\omega^2}{2} = \epsilon\hbar\omega/4$, and $V_{2o} = \frac{1}{2}\epsilon m\omega^2\langle 2|x^2|0\rangle = c_2\epsilon m\omega^2/2 = \frac{\hbar\sqrt{2}}{2m\omega}\frac{\epsilon m\omega^2}{2} = \epsilon\hbar\omega/2\sqrt{2}$. All other $V_{no} = 0$.