

Solutions to Final Take-home Exam
Physics 623, Spring 2009, O.W. Greenberg

(20 points) 1. Use first-order time-dependent perturbation theory to calculate the probability that an harmonic oscillator in its ground state in the infinite past ends
 (a) in its first excited state in the infinite future if it is excited by a time-dependent *force* (not potential)

$$F(t) = \frac{F_0\tau/\omega}{\tau^2 + t^2}. \quad (1)$$

- (b) in its second excited state over the same time interval.
 (c) What is the lowest order of time-dependent perturbation theory for which the oscillator will end in its n th excited state?

Solution

(a) We take the potential to be $-xF(t)$. First-order perturbation theory gives

$$c_n^{(1)}(t) = (-i/\hbar) \int_{t_0}^t \exp(i\omega_{ni}t') V_{ni}(t') dt'. \quad (2)$$

For our case, $t_0 = -\infty$, $t = \infty$, $\omega_{ni} = (E_n - E_i)/\hbar$, and $E_i = E_0 = \hbar\omega/2$, $E_n = E_1 = 3\hbar\omega/2$, so

$$c_i^{(1)}(\infty) = \frac{iF\tau}{\hbar\omega} \langle 1|x|0\rangle \int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{\tau^2 + t^2} dt \quad (3)$$

To evaluate the integral, we note that if $t \rightarrow t + i\epsilon$ the exponential factor is damped for $\epsilon > 0$, so we complete the contour in the upper half plane and use the residue at $\tau = it$ to find

$$\int_{-\infty}^{\infty} \frac{\exp(i\omega t)}{\tau^2 + t^2} dt = \frac{2\pi i}{2i\tau} \exp(-\omega\tau). \quad (4)$$

The matrix element is $\langle 1|x|0\rangle = \sqrt{\hbar/2m\omega}$. The amplitude becomes

$$c_i^{(1)}(\infty) = \frac{i\pi F}{\sqrt{2m\hbar\omega^3}} e^{-\omega\tau}. \quad (5)$$

The probability is

$$|c_i^{(1)}(\infty)|^2 = \frac{(\pi F)^2}{2m\hbar\omega^3} e^{-2\omega\tau}. \quad (6)$$

(b) The operator x changes the state of the oscillator by 1, so the second excited state is not reached in 1st order.

(c) The n th excited state is first excited in n th order.

(30 points) 2. The d -wave ($l = 2$) phase shift δ_2 increases through $\pi/2$ as the energy increases through $E = E_{res}$.

(a) What is the maximum d -wave cross section at E_{res} ?

(b) Give the most convenient parametrization you can find for $\delta_2(E)$ near E_{res} .

(c) The scattering amplitude for all the other partial waves, none of which are resonant near E_{res} , is $f_{nonres}(\theta)$. Express the total $f(\theta)$ in terms of the width and energy of the d -wave resonance, the mass of the scattering system m , Planck's constant and $f_{nonres}(\theta)$.

(d) Find the cross section for elastic scattering as a function of energy by integrating the differential cross section over solid angle.

(e) Find the total cross section of elastic scattering as a function of energy using the optical theorem.

Solution

(a) I should have stated that we assume the potential is spherically symmetric in order to use partial waves. The total elastic cross section in terms of partial waves is

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_l (2l + 1) \sin^2 \delta_l. \quad (7)$$

At resonance $\delta_l = \pi/2$, so for a d -wave, $l = 2$, at resonance the maximum cross section is

$$\sigma_2 = \frac{20\pi}{k_{res}^2} = \frac{10\pi\hbar^2}{mE_{res}} \quad (8)$$

You don't have to replace k_{res} using $E_{res} = \hbar^2 k_{res}^2 / 2m$. Either answer will do.

(b) Sakurai gives $\cot \delta \approx (-2/\Gamma)(E - E_{res})$ near E_{res} . So

$$\delta \approx \frac{\pi}{2} + \frac{2}{\Gamma}(E - E_{res}) \quad (9)$$

near E_{res} .

(c) The general expression for the elastic scattering amplitude in terms of partial waves is

$$f(\theta) = \sum_l (2l+1) f_l P_l(\cos \theta) \quad (10)$$

where there are various equivalent formulas for f_l . The most convenient one here is

$$f_l = \frac{1}{k \cot \delta_l - ik} \quad (11)$$

Then

$$f(\theta) = \frac{1}{k(-\frac{2}{\Gamma}(E - E_{res}) - i)} P_2(\theta) + f_{nonres}(\theta) \quad (12)$$

(d) The total elastic cross section is

$$\sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f(\theta)|^2 d\Omega \quad (13)$$

We use the orthogonality of the Legendre polynomials,

$$\int P_l(\cos \theta) P_{l'}(\cos \theta) d\Omega = \frac{2}{2l+1} \delta_{l,l'}, \quad (14)$$

so that the cross terms vanish and

$$\sigma_{tot} = \frac{10\pi\hbar^2}{mE} \frac{\frac{\Gamma^2}{2}}{(E - E_{res})^2 + \frac{\Gamma^2}{2}} + \int |f^{nonres}(\theta)|^2 d\Omega \quad (15)$$

(e) The optical theorem is $\sigma_{tot} = (4\pi/k) \text{Im} f(0)$.

$$\text{Im} f(0) = \frac{5\hbar}{\sqrt{2mE}} \frac{1}{-\frac{2}{\Gamma}(E - E_{res}) - i} + \text{Im} f_{nonres}(0) \quad (16)$$

using $P_l(0) = 1$. Then we get the same result as in (d), so the optical theorem checks.

It is not part of the problem, but note that

$$\sigma_l = \int (2l+1)^2 |f_l(\theta)|^2 d\Omega = 2\pi \frac{2}{2l+1} (2l+1)^2 \frac{\sin^2 \delta_l}{k^2} = \frac{4\pi(2l+1)\sin^2 \delta_l}{k^2} \quad (17)$$

and

$$\frac{4\pi}{k} (2l+1) \text{Im} f_l(0) = \frac{4\pi(2l+1)\sin^2 \delta_l}{k^2} \quad (18)$$

so (at least for elastic scattering) the optical theorem holds for each partial wave separately.

(25 points) 3. (This problem teaches you the relation between the Lorentz group, $SO(1,3)$ and its covering group, $SL(2,C)$.) Define a correspondence

$$(x^0, x^1, x^2, x^3) \leftrightarrow X = \sum_{\mu=0}^3 x^\mu \sigma_\mu \quad (19)$$

where the x^μ are real, $\sigma_0 = 1$ (the unit 2x2 matrix) and the σ_i are the Pauli matrices, $i = 1, 2, 3$.

- (a) What is the condition on X that the x^μ are real?
- (b) Find a formula for x^μ in terms of X .
- (c) Find $\det X$ in terms of the x^μ .
- (d) Let

$$X' = AXA^\dagger, \quad \det A = 1 \quad (20)$$

where A is a complex 2x2 matrix, of determinant one, as stated above, A^\dagger is the hermitian adjoint of A .

- (e) Prove that the x'^μ associated with X' is a Lorentz transform of the x^μ associated with X .
- (f) Which A 's can give the same X' ?
- (g) If we recognize that the Lorentz group has 4 disconnected components depending on whether there are no inversions, the space coordinates are inverted, the time coordinate is inverted, or the space-time coordinates are all inverted, prove which components can and cannot be related by the construction with A above.

Solution

(a) If we don't require x^μ to be real, we can find a condition on X such that the x^μ are real. Since the σ_μ 's are hermitian, $X^\dagger = \sum_{\mu=0}^3 x^{\mu*} \sigma_\mu$. Then the condition on X that x^μ are real is that X is hermitian.

(b) Since $tr(\sigma_\mu \sigma_\mu) = 2\delta_{\mu\nu}$, $x^\mu = (1/2)tr(\sigma_\mu X)$.

(c)

$$\det X = \det \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} = x^2 \quad (21)$$

(d) Not a question.

(e) Since $\det A = \det A^\dagger = 1$, $\det X^\dagger = \det X$ and $x'^2 = x^2$, which is the condition for a Lorentz transformation.

(f) If there is another matrix, call it B to save writing primes, that gives the same X' , then $BXB^\dagger = AXA^\dagger$ so $A^{-1}BX = X(A^{-1}B)^\dagger$. Now require, which I should have stated, that this be true for all hermitian X . Since $A^{-1}B = \sum_0^3 \alpha_\mu \sigma_\mu$, which is true for any 2x2 matrix, the trace argument shows that the α_μ are real. Then $A^{-1}B$ must commute with all 4 σ_μ 's, so $A^{-1}B$ is a multiple of the unit matrix, $\lambda 1$, with $\det = 1$, so $\lambda^2 = 1$ and only A and $-A$ relate X and X' for all X .

(g) To invert x^0 , we need $A\sigma_0 A^\dagger = -A$ or $AA^\dagger = -1$, since $\sigma_0 = 1$. This is impossible, since $tr AA^\dagger > 0$ and $tr -1 < 0$. Thus we cannot invert the time. To invert the space coordinates we need $A\sigma_i A^\dagger = \sigma_i = 1$, or A unitary, from just above, and $A\sigma_i = -\sigma_i A$, for all three σ_i 's. But no matrix anticommutes with all three σ_i 's. We can invert two space coordinates, which corresponds to a rotation in a plane by π , but not one or three space coordinates. Note that (just stating this) if we allow complex Lorentz transformations using $X' = AXB^\dagger$, with $\det A = 1$ and $\det B = 1$, then we can connect x and $-x$ continuously.

(25 points) 4. Calculate the energy and magnetic moment of a positive energy spin-1/2 particle of mass m and charge q in various external momentum-dependent potentials $V(|\mathbf{p}|)\Gamma$ using the Dirac equation in momentum space. Define the magnetic moment as the $|\mathbf{p}| \rightarrow 0$ limit of the expression for the magnetic moment as a function of $|\mathbf{p}|$. The potentials are

(a) $\Gamma = 1$, a scalar potential (that enters the Dirac equation like a mass term)

- (b) $\Gamma = \gamma^0$, a vector potential (that enters the Dirac equation like the zero component of a vector)
- (c) $\Gamma = \gamma^5$, a pseudoscalar potential (that enters the Dirac equation like γ^5) and
- (d) $\Gamma = i\gamma^0\gamma^5$, a pseudovector potential (that enters the Dirac equation like $i\gamma^0\gamma^5$).
- (e) What conditions do you have to impose on $V(|\mathbf{p}|)$ in the pseudovector case to get a finite $|\mathbf{p}| \rightarrow 0$ limit?

Solution

In each case consider $[c(\not{p} - (e/c)\not{A}) - V(|\mathbf{p}|) - mc^2]\psi(p) = 0$. The noncovariant form of the Dirac equation is convenient, so multiply from the left by β . You can do the full calculation to find the coefficient of the $\boldsymbol{\Sigma} \cdot \mathbf{B}$ term in the energy. An easier way to get the $\mathbf{p} \rightarrow 0$ limit is to argue

- (a) for the scalar potential that adds to the mass, the mass gets changed so you add a term to the usual mass and $E = \sqrt{(c\mathbf{p})^2 + (mc^2 + V(|\mathbf{p}|))^2}$. The magnetic moment just gets the new effective mass;

$$\mu = \frac{q\hbar}{2(m + (V(0)/c^2))c}. \quad (22)$$

- (b) for the vector potential that adds to the $p^0\gamma^0$ term the energy gets changed, so you add a term to the usual energy and $E = \sqrt{(c\mathbf{p})^2 + (mc^2)^2} + V(|\mathbf{p}|)$. Since the magnetic moment comes from the mass and the $i \neq j$ terms in $\gamma^i\gamma^j$, a γ^0 term will not change the magnetic moment, so $\mu = q\hbar/(2mc)$;

- (c) for the pseudoscalar potential both the mass and the energy change. Use

$$[E - c\mathbf{p} \cdot \boldsymbol{\alpha} - mc^2\beta - \beta\gamma^5V(|\mathbf{p}|)]\psi = 0 \quad (23)$$

Put this into 2x2 form,

$$\begin{pmatrix} E - mc^2 & -c\boldsymbol{\sigma} \cdot \mathbf{p} - V \\ -c\boldsymbol{\sigma} \cdot \mathbf{p} + V & E + mc^2 \end{pmatrix} = 0 \quad (24)$$

The determinant gives $E = \sqrt{(c\mathbf{p})^2 + (mc^2)^2} - V(|\mathbf{p}|)$. Now the effective mass is $\sqrt{m^2 - (V(0)/c^2)^2}$, so the magnetic moment is $\mu = q\hbar/(2\sqrt{(mc)^2 - (V(0)/c)^2})$. You need $V(0) \leq mc^2$ for the magnetic moment to be real.

- (d) For the pseudovector, there should not have been an i which makes the potential nonhermitian. You will get credit either using the i or removing the i .

The Dirac equation is (without the i),

$$[E - c(\boldsymbol{\alpha} \cdot \mathbf{p} - (e/c)\mathbf{A}) - mc^2\beta - \beta\gamma^5V(|\mathbf{p}|)]\psi = 0 \quad (25)$$

Put this into 2x2 form,

$$\begin{pmatrix} E - mc^2 & -c(\boldsymbol{\sigma} \cdot (\mathbf{p} - (e/c)\mathbf{A}) - V) \\ -c(\boldsymbol{\sigma} \cdot (\mathbf{p} - (e/c)\mathbf{A}) - V) & E + mc^2 \end{pmatrix} = 0 \quad (26)$$

When this acts on the column vector (ψ_1, ψ_2) we get

$$(E + mc^2)\psi_2 = [c\boldsymbol{\sigma} \cdot (\mathbf{p} - (q/c)\mathbf{A}) + V]\psi_1 \quad (27)$$

and

$$(E - mc^2)\psi_1 = [c\boldsymbol{\sigma} \cdot (\mathbf{p} - (q/c)\mathbf{A}) + V]\psi_2 \quad (28)$$

The equation for the large components ψ_1 becomes, keeping relevant terms,

$$(E - mc^2)\psi_1 = \left[\frac{(c\mathbf{p})^2 + V^2}{E + mc^2} - (q/c) \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{E + mc^2} + 2V \frac{\boldsymbol{\sigma} \cdot \mathbf{B}}{E + m} \right] \psi_1 \quad (29)$$

We introduce helicity eigenstates using

$$\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{|\mathbf{p}|}\psi_1 = h\psi_1 \quad (30)$$

The relevant terms in the equation for the energy are

$$E = \sqrt{(hc|\mathbf{p}| + V(|\mathbf{p}|))^2 + (mc^2)^2} \quad (31)$$

and the magnetic moment is

$$\mu = \frac{q\hbar}{\sqrt{m^2 + (V(0)/c^2)^2}c} \quad (32)$$