

Solutions for Take-home Exam for Physics 623, Spring 2009

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Please write each problem on a separate set of pages, stapled separately, so that the problems can be graded separately. Of course your name must be written in large legible letters on each problem.

1. A system has 3 degenerate low energy states coupled to a high energy state. The projector on the degenerate states is P_0 , the one on the high energy state is P_1 . Take the energy of the low energy states to be 0 and the energy of the high energy state to be E . Let the coupling of each of the low energy states to the high energy state to be V .

$$H = H_0 + H_I. \quad (1)$$

$$H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E \end{pmatrix} \quad (2)$$

$$H_I = \begin{pmatrix} 0 & 0 & 0 & V \\ 0 & 0 & 0 & V \\ 0 & 0 & 0 & V \\ V & V & V & 0 \end{pmatrix} \quad (3)$$

$$H = \begin{pmatrix} 0 & 0 & 0 & V \\ 0 & 0 & 0 & V \\ 0 & 0 & 0 & V \\ V & V & V & E \end{pmatrix} \quad (4)$$

(a) Construct an effective Hamiltonian in lowest nontrivial order in V/E

(a) This is a problem in degenerate perturbation theory with “distant” perturbation. Use (5.2.12) of Sakurai with V replaced by H_I . The effective Hamiltonian is

$$H_{eff} = P_0 H_0 P_0 + P_0 H_I P_1 (E_D^{(0)} - H_0)^{-1} P_1 H_I P_0. \quad (5)$$

The one tricky thing is the factor $(E_D^{(0)} - H_0)^{-1}$ where $E_D^{(0)} = 0$. There would be a zero in the denominator from H_0 , but it is sandwiched between P_1 projectors that project out any divergent factors. The result of the matrix calculation is a 4x4 matrix with zeroes outside of the 3x3 sector of the degenerate subspace with all 1's in the 3x3 subspace and coefficient $-V^2/E$. So the effective Hamiltonian in D that acts on the 3-dimensional subspace of the low energy states is

$$-\frac{V^2}{E} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad (6)$$

(b) Find the eigenstates of the effective Hamiltonian and choose a basis in which the states belong to specific irreducible representations of the symmetric group, S_3 .

(b and c) We can solve the secular equation, but it is simpler to see that the eigenstates will be in the totally symmetric representation of S_3 with nonzero eigenvalue and two eigenstates will be in the mixed representation of S_3 with eigenvalues 0. Thus in the symmetric representation of S_3 , the (unnormalized) eigenstate is $v_s = (1, 1, 1)$, and in the mixed representation of S_3 , so (unnormalized) eigenstates are linear combinations of $v_{m1} = (1, -1, 0)$ and $v_{m2} = (1, 1, -2)$. Acting with H_{eff} on these vectors (as vertical vectors) we find the eigenvalues are (in the same order) $-\frac{3V^2}{E}$, 0, 0. Since the trace is invariant the 4th eigenvalue of the full Hamiltonian must be $E + \frac{3V^2}{E}$.

(c) Find the eigenvalues of these states.

Given in the solution for (b).

(d) Redo the problem exactly as a 4-dimensional problem and compare the exact results with the results above.

(d) We check that the two zero eigenvectors are now linear combinations of $(1, -1, 0, 0)$ and $(1, 1, -2, 0)$. The other two eigenvectors will have most weight

either in the degenerate or the 4th state, so try $(1, 1, 1, x)$ and let the 4x4 matrix act on it. Get $(Vx, Vx, Vx, 3V + Ex)$. For this to be an eigenvector need $3V + Ex = Vx^2$. So get two roots, $x = (1/2V)(E \pm \sqrt{E^2 + 12V^2})$. The eigenvalues are $Vx = (1/2)(E \pm \sqrt{E^2 + 12V^2})$. In lowest order in V/E these are $E + (3V^2/E)$ and $-(3V^2)/E$ in agreement with the H_{eff} analysis. Again we can also get these results by solving the secular equation, but guessing as we did is easier.

2. A system has two levels of energy $E_1 = 0$ and $E_2 > 0$. The levels are coupled by purely off-diagonal oscillatory interactions, $V_{12} = \gamma \exp(i\omega t)$, $V_{21} = \gamma \exp(-i\omega t)$, γ, ω real.

(a) Assume only the upper level is populated at $t = 0$. Find the populations of the two states as a function of time by an exact calculation.

(a) Work in the interaction picture. $H_0 = E_2|2\rangle\langle 2|$.

$V(t) = \gamma \exp(i\omega t)|1\rangle\langle 2| + \gamma \exp(-i\omega t)|2\rangle\langle 1|$. The Schrödinger equation is

$$i\hbar \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} 0 & \gamma \exp(i\theta t) \\ \gamma \exp(i\theta t) & 0 \end{pmatrix} \quad (7)$$

$$\theta = \omega - \omega_{21} \quad \hbar\omega_{21} = E_2.$$

Convert a pair of first-order differential equations to a second-order one. Take the time derivative of both sides. Use the original first-order equations to express the zeroth- and first-order derivatives in terms of the first- and zeroth-order terms and get a diagonal pair of second-order equations,

$$i\hbar \begin{pmatrix} \ddot{c}_1 \\ \ddot{c}_2 \end{pmatrix} + i\theta\hbar \begin{pmatrix} \dot{c}_1 \\ -\dot{c}_2 \end{pmatrix} + (i\gamma^2/\hbar) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (8)$$

Work on the bottom equation

$$\ddot{c}_2 + i\theta\dot{c}_2 + (\gamma/\hbar)^2 c_2 = 0 \quad (9)$$

Try $c_2(t) \propto \exp(i\phi t)$. Find

$$\phi = -\theta/2 \pm \Omega \quad (10)$$

$$\theta = \omega - \omega_{21}, \quad \Omega = \sqrt{(\theta/2)^2 + (\gamma/\hbar)^2}.$$

$$c_2(t) = e^{-i\theta t/2} [A \cos \Omega t + B \sin \Omega t] \quad (11)$$

$$\begin{aligned}
c_1(t) &= \frac{i\hbar}{\gamma} e^{i\theta t} \dot{c}_2(t) \\
&= \frac{i\hbar}{\gamma} e^{i\theta t/2} \left\{ \left[\frac{-i\theta}{2} \cos \Omega t - \Omega \sin \Omega t \right] A + \left[\frac{-i\theta}{2} \sin \Omega t + \Omega \cos \Omega t \right] B \right\} \quad (12)
\end{aligned}$$

The initial conditions give $A = 1$, $B = i\theta/2\Omega$. Then

$$c_2(t) = e^{-i\theta t/2} \left[\cos \Omega t + \frac{i\theta}{2\Omega} \sin \Omega t \right] \quad (13)$$

$$c_1(t) = \frac{i\hbar}{\gamma} e^{i\theta t/2} \left(\frac{\theta^2}{4\Omega} - \Omega \right) \sin \Omega t \quad (14)$$

- (b) Write the parameter for the oscillatory time dependence of the two states.
- (b) The time dependence is sinusoidal with angular frequency Ω .
- (c) Find the condition for maximum amplitude of the oscillations. What do we call this condition?
- (c) The maximum amplitude occurs when $\theta = \omega - \omega_{21} = 0$; i.e. when $\omega = \omega_{21}$, at resonance.
- (d) Redo this problem using time-dependent perturbation theory in lowest nontrivial order.
- (d) Use $c_2^{(0)}(t) = 1$, $c_1^0(T) = 0$, initial conditions.

$$c_2^{(1)}(t) = \frac{-i}{\hbar} \int_0^t e^{-i\omega_{21}t'} \langle 2|V(t')|2 \rangle dt' = 0 \quad (15)$$

$$c_1^{(1)}(t) = \frac{-i}{\hbar} \int_0^t e^{-i\omega_{21}t'} \langle 1|V(t')|2 \rangle dt' = \frac{-\gamma}{\hbar} (e^{i\theta t} - 1) \quad (16)$$

Compare with the order γ limits of the exact results. A simple calculation gives the same result, 0 for $c_2^{(1)}(t)$. The result for $c_1^{(1)}(t)$ requires keeping the order γ^2 term in $\Omega \approx \theta/2 + \gamma^2/(\hbar^2\theta)$. A mildly tedious calculation then gives agreement with order γ perturbation theory for $c_1^{(1)}(t)$

- (e) Write several physical problems to which this example is relevant.
- (e) This example is relevant to many two-state problems, such as molecular beam studies to find the magnetic moments of nuclei, masers, quantum optics, optical pumping, and the study of “negative temperatures.”

3.(a) Construct a trial wave function that obeys the Heisenberg-Weyl inequality

$$T_\psi \langle |x|^2 \rangle_\psi \geq 9/4, \quad (17)$$

where $T_\psi = \int [\nabla \psi(x)]^2 dx$, $\langle |x|^2 \rangle_\psi = \int |x|^2 [\psi(x)]^2 dx$, subject to $\int [\psi(x)]^2 dx = 1$, and yet allows the energy of a single-electron atom to be unbounded below.

(a) Following Lieb in RMP, choose ψ to be concentrated inside a radius R near the origin with probability $1/2$ and in a thin shell at distance L away from the origin also with probability $1/2$. The expectation of the energy is,

$$\langle H \rangle = T_\psi - Z \langle |\mathbf{x}|^{-1} \rangle_\psi \quad (18)$$

(Following Lieb, the x 's are three dimensional vectors.) Note that the Sobolev inequality depends on the dimension of space, unlike the Heisenberg-Weyl inequality. We would like to get T_ψ very large from the Heisenberg uncertainty principle inequality so as to prevent the energy getting very negative.

Unfortunately, the Heisenberg uncertainty principle factor goes in the denominator, so we get

$$\langle H \rangle \geq (9/4)(2/L^2) - (Z^2/2R) \rightarrow -\infty \quad (19)$$

for L large enough and R small enough, even though we used the Heisenberg uncertainty principle.

(b) Use the Sobolev inequality

$$T_\psi \geq K_s \left[\int [\psi(x)]^6 dx \right]^{1/3} \quad (20)$$

to find a finite lower bound for the energy of a single-electron atom.

K_s is a positive number whose value is not needed.

(b) Again following Lieb, we want to minimize the expectation value of the Hamiltonian,

$$\langle H \rangle = T_\psi - Z \int |x|^{-1} \rho dx \equiv h(\rho) \quad (21)$$

subject to $\int \rho dx = 1$. We defined $\rho = |\psi|^2$. Put in the constraint as a Lagrange multiplier by adding $\lambda(\int \rho dx - 1)$ to $h(\rho)$.

$$\frac{\delta}{\delta \rho(x)} [h(\rho) + \lambda(\int \rho dx - 1)] = K_s \left[\int \rho^3 dx \right]^{-2/3} \rho(x) - \frac{Z}{|x|} + \lambda = 0 \quad (22)$$

$$\frac{\delta}{\delta\lambda} \implies \int \rho dx = 1 \quad (23)$$

We find

$$\rho(x) = \sqrt{\frac{Z}{K_s}} [\int \rho^3 dx]^{1/3} \sqrt{\frac{1}{|x|} - \frac{\lambda}{Z}} \quad (24)$$

Since ρ must be positive or zero, we need $\frac{1}{|x|} - \frac{\lambda}{Z} \geq 0$, or $|x| \leq \frac{Z}{\lambda} \equiv R$. Then

$$\rho_{min} = \begin{cases} \alpha \sqrt{\frac{1}{|x|} - \frac{1}{R}}, & |x| \leq R \\ 0, & |x| > R \end{cases} \quad (25)$$

where from Eq.(24) $\alpha = \sqrt{\frac{Z}{K_s}} [\int \rho^3 dx]^{1/3}$. Mathematica gives this as a finite expression. The value of R follows from $\int \rho dx = 1$. Mathematica also gives this as a finite expression, $\alpha = (\pi^2/4)^{2/3} (RZ^2/K_s)$. Inserting ρ_{min} into $h(\rho)$ then gives a finite lower bound for the energy.

References: class notes and E.H. Lieb, Rev. Mod. Phys. 48, 553 (1976).

4. Consider 3 quark states of u, d, s quarks of spin 1/2 in the $SU(6)$ state appropriate for fermi quarks.

(a) In what irreducible of $SU(6)$ will they be?

Each quark is a 6 of $SU(6)$ so the 3-quark states are in

$6 \times 6 \times 6 \rightarrow 56 + 70 + 70 + 20$. These irreducibles are in the Young tableau

$(3), (2, 1), (2, 1), (1, 1, 1)$. Fermions must be in the antisymmetric representation, $20 = (1, 1, 1)$.

(b) Decompose this irreducible into $(SU(3)_f, SU(2)_s)$ states of $SU(3)$ and spin.

Label the $SU(3)$ states by their multiplicity; label the $SU(2)_s$ states by their spin.

Decompose the $SU(3)$ states into (I=isospin, Y=hypercharge) states.

The 2-quark states are in 6×6 which belong to $21 = (2)$ and $15 = (1, 1)$. For fermi quarks we want the 15. Since one quark, as a 6 of $SU(6)$, decomposes into $(3, 1/2)$ under $SU(6) \rightarrow (SU(3) \times SU(2)_s)$, two quarks decompose into

$(3, 1/2) \times (3, 1/2) \rightarrow ((6, 1) = (2, 2)) + ((6, 0) = (2, (1, 1))) + ((3^*, 1) =$

$((1, 1), 2)) + ((3^*, 0) = ((1, 1), (1, 1)))$ under $SU(6) \rightarrow (SU(3) \times SU(2)_s)$ For fermi

quarks we want the antisymmetric part of 6×6 which is 15 in $SU(6)$, so we want

$((6, 0) = (2, (1, 1))) + ((3^*, 1 = ((1, 1), 2))$. The multiplicities add to 15 as they

should. The 3-quark fermi state is in the antisymmetric part of 15×6 which by similar arguments is $((8 = (2, 1), 1/2 = (2, 1))) + ((1 = (1, 1, 1), 3/2 = 3))$. So the fermi 3-quark states are in an octet of spin 1/2 that looks like the nucleon octet and a singlet of spin 3/2. The nucleon octet decomposes into

$(1/2, 1) + (1, 0) + (0, 0) + (1/2, -1)$ which look like

$(p, n) + (\Sigma^+, \Sigma^0, \Sigma^-) + (\Lambda^0) + (\Xi^0, \Xi^-)$.

(c) Find the magnetic moments of the proton and neutron states that occur assuming the quarks have point-like magnetic moments given by $2\mu_0 \sum_q Q_q S_q$, where μ_0 is a magnetic moment factor that gives the scale, Q_q is the charge of the quark, and S_q is the z-component of the spin of the quark.

Hint: you must construct the properly symmetrized or antisymmetrized state for each particle in terms of its constituent quarks.

The particles that look like the proton and neutron but with antisymmetric fermi quarks must be in the states

$$|p \uparrow\rangle = |u \uparrow \ u \downarrow \ d \uparrow\rangle, \quad (26)$$

$$|n \uparrow\rangle = |d \uparrow \ d \downarrow \ u \uparrow\rangle. \quad (27)$$

For the “proton” the contributions of the u quarks cancel and the d quark gives $\mu_p = -\mu_0/3$. For the “neutron” the contributions of the d quarks cancel and the u quark gives $\mu_n = 2\mu_0/3$. The ratio $\mu_p/\mu_n = -1/2$; the observed ratio is close to $-3/2$. The magnetic moments even point the wrong way.

Note that using fermi operators to create the states automatically takes care of the necessary antisymmetrization. Using wave functions is more complicated. ($u \uparrow$ stands for the fermi creation operator that makes an up-spin u-quark, etc.)

Reference: Georgi, Lie Algebras ... on reserve in Engineering Library.