Lecture 9. Momentum Representation, Change Basis, More Examples, Wednesday, Sept. 21

Work out the momentum operator in the x-representation following the textbook.

The eigenvalues of $\hat{p}$ are also continuous and span a one-dimensional real axis. Eigenstates $|p\rangle$ can be chosen as a basis in the Hilbert space,

$$\langle p|p'\rangle = \lambda\delta(p-p') \ ,$$
$$\int \frac{dp}{\lambda}|p\rangle\langle p| = 1 \ , \tag{85}$$

where λ can be chosen as anything. It is 1 in JJS; I usually choose $2\pi\hbar$.

The momentum eigenstates can be expressed in $|x\rangle$ basis. The eigen-equation in x-rep is well-known,

$$-i\hbar\frac{d}{dx}\langle x|p\rangle = p\langle x|p\rangle \tag{86}$$

and the solution is

$$\langle x|p\rangle = e^{ixp/\hbar} \ . \tag{87}$$

We note that

$$\int_{-\infty}^{\infty} e^{ikx}dx = 2\pi\delta(x) \ . \tag{88}$$

Momentum representation: choose $|p\rangle$ as a basis, we have the momentum space wave function

$$\psi(p) = \langle p|\psi\rangle \ . \tag{89}$$

It can be shown that the momentum space wave function is related to the coordinate space wave function by simple Fourier transformation.

The position operator in the momentum representation: $\hat{x}$ is the generator of translation in the momentum space,

$$\hat{x} = i\hbar\frac{d}{dp} \ . \tag{90}$$

Gaussian wave packet.

$$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}\sqrt{d}}\exp\Big(ikx - x^2/2d^2\Big) \tag{91}$$

Plot its probability density. Calculate the expectation value $\langle\alpha|x|\alpha\rangle = 0$ because of the symmetry. On the other hand $\langle\alpha|x^2|\alpha\rangle = d^2/2$, so one has, $\Delta x = d/\sqrt{2}$. Likewise, $\Delta p = \hbar/\sqrt{2}d$. Therefore, $\Delta x \Delta p = \hbar/2$.

Review harmonic oscillator, one-dimensional square well potential problems.

Change of Basis: Consider two bases, $|i\rangle$ and $|j'\rangle$. The two bases are related by a unitary transformation

$$|i'\rangle = U|i\rangle \tag{92}$$

where $U^\dagger U = UU^\dagger = 1$. If we insert $\sum_j |j\rangle\langle j| = 1$, then we have,

$$|i'\rangle = \sum_j |j\rangle\langle j|U|i\rangle = \sum_j |j\rangle U_{ji} \tag{93}$$

where $U_{ij} = \langle i|U|j\rangle$. We can write the above equation in a matrix form,

$$(|1'\rangle, |2'\rangle, ...) = (|1\rangle, |2\rangle, ...) \begin{pmatrix} U_{11} & U_{12} & ... \\ U_{21} & U_{22} & ... \\ ... & ... & ... \end{pmatrix} \tag{94}$$

where the basis vectors appear as a row matrix.

Suppose a vector $|\psi\rangle = \sum_i c_i|i\rangle$ in the old basis, and we can represent $|\psi\rangle$ as a column matrix of c_i, or

$$|\psi\rangle = (|1\rangle, |2\rangle, ...) \begin{pmatrix} c_1 \\ c_2 \\ ... \end{pmatrix} \tag{95}$$

The same vector can be expressed as $|\psi\rangle = \sum_i c_i'|i'\rangle$ in the new basis. Then it is easy to see that

$$c_i' = \sum_j U_{ij}^{-1} c_j \ , \tag{96}$$

or we can replace the U^{-1} by $U^\dagger$ because of the unitarity.

In some textbooks, the U here is denoted by U^{-1}.

Consider also an operator $O = \sum_{ij} O_{ij}|i\rangle\langle j|$, which can be written also as a matrix form,

$$O = (|1\rangle, |2\rangle, ...) \begin{pmatrix} O_{11} & O_{12} & ... \\ O_{21} & O_{22} & ... \\ ... & ... & ... \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \\ ... \end{pmatrix} \ , \tag{97}$$

From this, it is easy to show that

$$O'_{ij} = \sum_{kl} U^\dagger_{ik} O_{kl} U_{lj} \ , \tag{98}$$

or simply $O' = U^\dagger O U$ in the matrix sense.

The trace of an matrix is independent of bases.

Suppose we have a matrix O, and we diagonalize it in the old basis $|i\rangle$. Suppose all eigenvectors are $|\lambda_n\rangle$. Then in the $|\lambda_n\rangle$ basis, the matrix is diagonal with eigenvalue λ_n. The transformation matrix from the old to the new basis is $|\lambda_i\rangle = U|i\rangle$. Thus we can write,

$$O = U^\dagger O_D U \ , \tag{99}$$

where O_D is the diagonal matrix, and U is a matrix whose columns are formed by eigenvectors.

Example of σ_y.

If two observables are related by unitary transformation A, $B = UAU^{-1}$, we say A and B are unitary equivalent observables. It is easy to see that A and B have exactly the same eigenvalues, and their eigenstates are unitary transformation of each other. S_x and S_y and S_z are unitary equivalent observables.

Discuss homework problems. Solve new problems, time permits.

Hints: 1.21: Start with the normalized wave function in the square well potential.

$$\psi_n = \sqrt{\frac{2}{a}} \sin x \frac{n\pi}{a} \tag{100}$$

where $n = 1, 2, ...$ In the x-representation, $p = -i\hbar d/dx$.

1.26: Consider S_x as the old basis. Diagonalize S_x in the old basis. Express U in the matrix form.

1.29: Consider 1D case, multi-D is easy because $[x_i, x_j] = 0$. Show it is true for a monomial p^n. Classical Poisson bracket definition (1.6.48).

1.33: a) Insert $\int dx |x\rangle\langle x| = 1$ and use $ixe^{ipx} = d(e^{ixp})/dp$ b) Take the matrix element of the exponential operator between $\langle x|$ and $|p\rangle$, and see what you get!