Lecture 28, More on Angular Momentum Coupling, Monday, Nov. 14

Wigner 3j symbol: Introduce

$$\langle j_1 m_1 j_2 m_2 | jm \rangle = (-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \tag{112}$$

The 3j-symbol has the following properties: if one make a even number of permutations of the columns, there is no change,

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} = \begin{pmatrix} j_2 & j & j_1 \\ m_2 & -m & m_1 \end{pmatrix} = \begin{pmatrix} j & j_1 & j_2 \\ m & m_1 & m_2 \end{pmatrix} \tag{113}$$

However, if one makes an odd number of permutations of the columns, the result changes by a phase factor $(-1)^{j_1+j_2+j}$.

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} = (-1)^{j_1+j_2+j}\begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & -m \end{pmatrix} \tag{114}$$

The same phase factor follows if one changes the signs of all the m's

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} = (-1)^{j_1+j_2+j}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \tag{115}$$

The 3j symbol make the three angular momenta in a equal position. In fact we could call j as j_3, and $-m$ as m_3.

Here are some useful formula for 3j symbols,

$$\begin{aligned}
\begin{pmatrix} j & -j & 0 \\ m & -m & 0 \end{pmatrix} &= (-1)^{j-m}\frac{1}{\sqrt{2j+1}} \\
\begin{pmatrix} j+\frac{1}{2} & j & \frac{1}{2} \\ m & -m-\frac{1}{2} & \frac{1}{2} \end{pmatrix} &= (-1)^{j-m-1/2}\sqrt{\frac{j-m+\frac{1}{2}}{(2j+1)(2j+2)}} \\
\begin{pmatrix} j & -j & 1 \\ m & -m & 0 \end{pmatrix} &= (-1)^{j-m}\frac{2m}{\sqrt{2j(2j+1)(2j+2)}}
\end{aligned} \tag{116}$$

More results on 3j symbols can be found in Landau & Lifshitz, Quantum Mechanics.

Using 3j symbols and their symmetry properties, one can find out what happens to the C.G. coefficients if one changes the order of coupling. Indeed,

$$\begin{aligned}\langle j_1 m_1 j_2 m_2 | jm \rangle &= (-1)^{j_1-j_2+m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix} \\ &= (-1)^{3j_1+j-j_2}(-1)^{j_2-j_1+m}\sqrt{2j+1}\begin{pmatrix} j_2 & j_1 & j \\ m_2 & m_1 & -m \end{pmatrix} \\ &= (-1)^{j-j_1-j_2}\langle j_2 m_2 j_1 m_1 | jm \rangle \end{aligned} \tag{117}$$

This symmetry relation is very useful for finding out the symmetry properties of wave functions.

Using the Wigner-Eckart theorem, one can prove the projection theorem that the matrix element of any vector operator V_q can be written as

$$\langle \alpha' jm' | V_q | \alpha, jm \rangle = \langle \alpha' jm | \vec{J} \cdot \vec{V} | \alpha jm \rangle \langle jm' | J_q | jm \rangle \frac{1}{\hbar^2 j(j+1)} \tag{118}$$

which has important applications in calculating the magnetic moments of atomic systems.

For example, if we want to calculate the matrix element of s_z in a state of ψ_{nljm}, we have,

$$\begin{aligned}\langle ljm | s_z | ljm \rangle &= \langle jm | J_z | jm \rangle \langle jm | \vec{j} \cdot \vec{s} | jm \rangle / j(j+1) \\ &= \frac{m}{2j(j+1)} \langle (\vec{j}^2 + \vec{s}^2 - (\vec{j} - \vec{s})^2) \rangle \\ &= \frac{m}{2j(j+1)} (j(j+1) + 3/4 - l(l+1)) \end{aligned} \tag{119}$$

More examples on the reduced matrix elements: Consider the scalar operator 1, we have

$$\langle j'm' | 1 | jm \rangle = \delta_{jj'} \delta mm' \frac{\langle j' || 1 || j \rangle}{\sqrt{2j+1}} \tag{120}$$

The matrix element on the left hand side is just $\delta_{jj'}\delta_{mm'}$, so we have,

$$\langle j' || 1 || j \rangle = \delta_{jj'} \sqrt{2j+1} \tag{121}$$

Second example, consider the matrix element of j_z,

$$\langle j'm' | j_z | jm \rangle = \langle jm10 | j'm' \rangle \frac{\langle j' || \vec{j} || j \rangle}{\sqrt{2j'+1}} \tag{122}$$

Using the CG coefficient which can be derived from the 3j symbol given above, one has,

$$\langle j'||\vec{j}||j\rangle = \delta_{jj'}\sqrt{j(j+1)(2j+1)} \tag{123}$$

Spherical tensors and Wigner-Eckart theorems are powerful tools for calculating matrix elements.